1 Integrale mit $ax + b \quad (a \neq 0)$

Hinweis: Im Sonderfall $b = 0$ erhält man Integrale von *Potenzen*, die mit Hilfe der Potenzregel der Integralrechnung elementar lösbar sind.

(1) $$\int (ax+b)^n \, dx = \frac{(ax+b)^{n+1}}{(n+1)\,a} \qquad (n \neq -1)$$

Fall $n = -1$: siehe Integral (2)

(2) $$\int \frac{dx}{ax+b} = \frac{1}{a} \cdot \ln|ax+b|$$

(3) $$\int x(ax+b)^n \, dx = \frac{(ax+b)^{n+2}}{(n+2)\,a^2} - \frac{b(ax+b)^{n+1}}{(n+1)\,a^2} \qquad (n \neq -1, -2)$$

Fall $n = -1, -2$: siehe Integral (4) bzw. (5)

(4) $$\int \frac{x\,dx}{ax+b} = \frac{x}{a} - \frac{b}{a^2} \cdot \ln|ax+b|$$

(5) $$\int \frac{x\,dx}{(ax+b)^2} = \frac{b}{a^2(ax+b)} + \frac{1}{a^2} \cdot \ln|ax+b|$$

(6) $$\int \frac{x\,dx}{(ax+b)^n} = -\frac{1}{(n-2)\,a^2(ax+b)^{n-2}} + \frac{b}{(n-1)\,a^2(ax+b)^{n-1}} \qquad (n \neq 1, 2)$$

Fall $n = 1, 2$: siehe Integral (4) bzw. (5)

(7) $$\int x^2(ax+b)^n \, dx = \frac{(ax+b)^{n+3}}{(n+3)\,a^3} - \frac{2b(ax+b)^{n+2}}{(n+2)\,a^3} + \frac{b^2(ax+b)^{n+1}}{(n+1)\,a^3} \qquad (n \neq -1, -2, -3)$$

Fall $n = -1, -2, -3$: siehe Integral (8), (9) bzw. (10)

(8) $$\int \frac{x^2\,dx}{ax+b} = \frac{(ax+b)^2}{2a^3} - \frac{2b(ax+b)}{a^3} + \frac{b^2}{a^3} \cdot \ln|ax+b|$$

(9) $$\int \frac{x^2\,dx}{(ax+b)^2} = \frac{ax+b}{a^3} - \frac{b^2}{a^3(ax+b)} - \frac{2b}{a^3} \cdot \ln|ax+b|$$

(10) $$\int \frac{x^2\,dx}{(ax+b)^3} = \frac{2b}{a^3(ax+b)} - \frac{b^2}{2a^3(ax+b)^2} + \frac{1}{a^3} \cdot \ln|ax+b|$$

(11) $$\int \frac{x^2\,dx}{(ax+b)^n} = -\frac{1}{(n-3)\,a^3(ax+b)^{n-3}} + \frac{2b}{(n-2)\,a^3(ax+b)^{n-2}} - \frac{b^2}{(n-1)\,a^3(ax+b)^{n-1}}$$

$(n \neq 1, 2, 3)$. Fall $n = 1, 2, 3$: siehe Integral (8), (9) bzw. (10)

(12) $$\int \frac{dx}{x(ax+b)} = -\frac{1}{b} \cdot \ln\left|\frac{ax+b}{x}\right|$$

(13) $$\int \frac{dx}{x(ax+b)^2} = \frac{1}{b(ax+b)} - \frac{1}{b^2} \cdot \ln\left|\frac{ax+b}{x}\right|$$

(14) $$\int \frac{dx}{x(ax+b)^3} = \frac{a^2x^2}{2b^3(ax+b)^2} - \frac{2ax}{b^3(ax+b)} - \frac{1}{b^3} \cdot \ln\left|\frac{ax+b}{x}\right|$$

(15) $$\int \frac{dx}{x^2(ax+b)} = -\frac{1}{bx} + \frac{a}{b^2} \cdot \ln\left|\frac{ax+b}{x}\right|$$

(16) $$\int \frac{dx}{x^2(ax+b)^2} = -\frac{a}{b^2(ax+b)} - \frac{1}{b^2x} + \frac{2a}{b^3} \cdot \ln\left|\frac{ax+b}{x}\right|$$

(17) $$\int \frac{dx}{x^3(ax+b)} = -\frac{(ax+b)^2}{2b^3x^2} + \frac{2a(ax+b)}{b^3x} - \frac{a^2}{b^3} \cdot \ln\left|\frac{ax+b}{x}\right|$$

(18) $$\int \frac{dx}{x^3(ax+b)^2} = -\frac{(ax+b)^2}{2b^4x^2} + \frac{3a(ax+b)}{b^4x} - \frac{a^3x}{b^4(ax+b)} - \frac{3a^2}{b^4} \cdot \ln\left|\frac{ax+b}{x}\right|$$

(19) $$\int x^m(ax+b)^n\,dx = \begin{cases} \dfrac{x^{m+1}(ax+b)^n}{m+n+1} + \dfrac{nb}{m+n+1} \cdot \displaystyle\int x^m(ax+b)^{n-1}\,dx & (m+n \neq -1) \\[2ex] \dfrac{x^m(ax+b)^{n+1}}{(m+n+1)a} - \dfrac{mb}{(m+n+1)a} \cdot \displaystyle\int x^{m-1}(ax+b)^n\,dx & (m+n \neq -1) \\[2ex] -\dfrac{x^{m+1}(ax+b)^{n+1}}{(n+1)b} + \dfrac{m+n+2}{(n+1)b} \cdot \displaystyle\int x^m(ax+b)^{n+1}\,dx & (n \neq -1) \end{cases}$$

2 Integrale mit $ax + b$ und $px + q$ $(a, p \neq 0)$

Abkürzung: $\boxed{\Delta = bp - aq}$

Hinweis: Es wird stets $\Delta \neq 0$ vorausgesetzt. Für $\Delta = 0$ ist $px + q = \dfrac{q}{b}(ax + b)$.
Die Integrale entsprechen dann dem Integraltyp aus Abschnitt 1.

(20) $$\int \frac{ax+b}{px+q}\,dx = \frac{ax}{p} + \frac{\Delta}{p^2} \cdot \ln|px+q|$$

(21) $$\int \frac{dx}{(ax+b)(px+q)} = \frac{1}{\Delta} \cdot \ln\left|\frac{px+q}{ax+b}\right|$$

(22) $$\int \frac{dx}{(ax+b)^2(px+q)} = \frac{1}{\Delta}\left[\frac{1}{ax+b} + \frac{p}{\Delta} \cdot \ln\left|\frac{px+q}{ax+b}\right|\right]$$

(23) $$\int \frac{dx}{(ax+b)^m(px+q)^n} = -\frac{1}{(n-1)\Delta}\left[\frac{1}{(ax+b)^{m-1}(px+q)^{n-1}} + \right.$$
$$\left. + (m+n-2)a \cdot \int \frac{dx}{(ax+b)^m(px+q)^{n-1}}\right] \qquad (n \neq 1)$$

Fall $n = 1$: siehe Integral (24)

(24) $\int \frac{(ax+b)^m}{px+q}\,dx = \frac{(ax+b)^m}{mp} + \frac{\Delta}{p} \cdot \int \frac{(ax+b)^{m-1}}{px+q}\,dx \qquad (m \neq 0)$

Fall $m = 0$: siehe Integral (2)

(25) $\int \frac{(ax+b)^m}{(px+q)^n}\,dx = -\frac{1}{(n-1)\,p}\left[\frac{(ax+b)^m}{(px+q)^{n-1}} - ma \cdot \int \frac{(ax+b)^{m-1}}{(px+q)^{n-1}}\,dx\right] \qquad (n \neq 1)$

Fall $n = 1$: siehe Integral (24)

(26) $\int \frac{x\,dx}{(ax+b)(px+q)} = \frac{1}{\Delta}\left[\frac{b}{a} \cdot \ln|ax+b| - \frac{q}{p} \cdot \ln|px+q|\right]$

(27) $\int \frac{x\,dx}{(ax+b)^2(px+q)} = \frac{1}{\Delta}\left[-\frac{b}{a(ax+b)} + \frac{q}{\Delta} \cdot \ln\left|\frac{ax+b}{px+q}\right|\right]$

(28) $\int \frac{x^2\,dx}{(ax+b)^2(px+q)} = \frac{b^2}{a^2\Delta(ax+b)} + \frac{1}{\Delta^2}\left[\frac{q^2}{p} \cdot \ln|px+q| + \frac{b(bp-2aq)}{a^2} \cdot \ln|ax+b|\right]$

3 Integrale mit $a^2 + x^2 \quad (a > 0)$

(29) $\int \frac{dx}{a^2+x^2} = \frac{1}{a} \cdot \arctan\left(\frac{x}{a}\right)$

(30) $\int \frac{dx}{(a^2+x^2)^2} = \frac{x}{2a^2(a^2+x^2)} + \frac{1}{2a^3} \cdot \arctan\left(\frac{x}{a}\right)$

(31) $\int \frac{dx}{(a^2+x^2)^n} = \frac{x}{2(n-1)\,a^2(a^2+x^2)^{n-1)}} + \frac{2n-3}{2(n-1)\,a^2} \cdot \int \frac{dx}{(a^2+x^2)^{n-1}} \qquad (n \neq 1)$

Fall $n = 1$: siehe Integral (29)

(32) $\int \frac{x\,dx}{a^2+x^2} = \frac{1}{2} \cdot \ln(a^2+x^2)$

(33) $\int \frac{x\,dx}{(a^2+x^2)^2} = -\frac{1}{2(a^2+x^2)}$

(34) $\int \frac{x\,dx}{(a^2+x^2)^n} = -\frac{1}{2(n-1)(a^2+x^2)^{n-1}} \qquad (n \neq 1)$

Fall $n = 1$: siehe Integral (32)

(35) $\int \frac{x^2\,dx}{a^2+x^2} = x - a \cdot \arctan\left(\frac{x}{a}\right)$

(36) $\int \frac{x^2\,dx}{(a^2+x^2)^2} = -\frac{x}{2(a^2+x^2)} + \frac{1}{2a} \cdot \arctan\left(\frac{x}{a}\right)$

(37)	$\int \frac{x^2\,dx}{(a^2+x^2)^n} = -\frac{x}{2(n-1)(x^2+a^2)^{n-1}} + \frac{1}{2(n-1)} \cdot \underbrace{\int \frac{dx}{(a^2+x^2)^{n-1}}}_{\text{Integral (31)}} \qquad (n \neq 1)$ Fall $n = 1$: siehe Integral (35)
(38)	$\int \frac{dx}{x(a^2+x^2)} = -\frac{1}{2a^2} \cdot \ln\left(\frac{a^2+x^2}{x^2}\right)$
(39)	$\int \frac{dx}{x(a^2+x^2)^2} = \frac{1}{2a^2(a^2+x^2)} - \frac{1}{2a^4} \cdot \ln\left(\frac{a^2+x^2}{x^2}\right)$
(40)	$\int \frac{dx}{x^2(a^2+x^2)} = -\frac{1}{a^2 x} - \frac{1}{a^3} \cdot \arctan\left(\frac{x}{a}\right)$
(41)	$\int \frac{dx}{x^2(a^2+x^2)^2} = -\frac{1}{a^4 x} - \frac{x}{2a^4(a^2+x^2)} - \frac{3}{2a^5} \cdot \arctan\left(\frac{x}{a}\right)$
(42)	$\int \frac{x^m\,dx}{(a^2+x^2)^n} = \int \frac{x^{m-2}\,dx}{(a^2+x^2)^{n-1}} - a^2 \cdot \int \frac{x^{m-2}\,dx}{(a^2+x^2)^n}$
(43)	$\int \frac{dx}{x^m(a^2+x^2)^n} = \frac{1}{a^2} \cdot \int \frac{dx}{x^m(a^2+x^2)^{n-1}} - \frac{1}{a^2} \cdot \int \frac{dx}{x^{m-2}(a^2+x^2)^n}$
(44)	$\int \frac{dx}{(px+q)(a^2+x^2)} = \frac{1}{a^2p^2+q^2}\left[\frac{p}{2} \cdot \ln\left(\frac{(px+q)^2}{a^2+x^2}\right) + \frac{q}{a} \cdot \arctan\left(\frac{x}{a}\right)\right] \qquad (p \neq 0)$
(45)	$\int \frac{x\,dx}{(px+q)(a^2+x^2)} = \frac{1}{2(a^2p^2+q^2)}\left[q \cdot \ln\left(\frac{a^2+x^2}{(px+q)^2}\right) + 2ap \cdot \arctan\left(\frac{x}{a}\right)\right] \qquad (p \neq 0)$

4 Integrale mit $a^2 - x^2 \quad (a > 0)$

Hinweis: Die in den nachfolgenden Integralformeln auftretende *logarithmische* Funktion $\ln\left|\frac{a+x}{a-x}\right|$ kann auch wie folgt durch *Areafunktionen* ersetzt werden:

$$\ln\left|\frac{a+x}{a-x}\right| = \begin{cases} \ln\left(\frac{a+x}{a-x}\right) = 2 \cdot \operatorname{artanh}\left(\frac{x}{a}\right) & \text{für } \lvert x\rvert < a \\ \ln\left(\frac{x+a}{x-a}\right) = 2 \cdot \operatorname{arcoth}\left(\frac{x}{a}\right) & \text{für } \lvert x\rvert > a \end{cases}$$

(46)	$\int \frac{dx}{a^2-x^2} = \frac{1}{2a} \cdot \ln\left|\frac{a+x}{a-x}\right| = \begin{cases} \frac{1}{a} \cdot \operatorname{artanh}\left(\frac{x}{a}\right) & \text{für } \lvert x\rvert < a \\ \frac{1}{a} \cdot \operatorname{arcoth}\left(\frac{x}{a}\right) & \text{für } \lvert x\rvert > a \end{cases}$
(47)	$\int \frac{dx}{(a^2-x^2)^2} = \frac{x}{2a^2(a^2-x^2)} + \frac{1}{4a^3} \cdot \ln\left|\frac{a+x}{a-x}\right|$

(48)	$\int \frac{dx}{(a^2 - x^2)^n} = \frac{x}{2(n-1)\,a^2\,(a^2 - x^2)^{n-1}} + \frac{2n-3}{2(n-1)\,a^2} \cdot \int \frac{dx}{(a^2 - x^2)^{n-1}} \qquad (n \neq 1)$ Fall $n = 1$: siehe Integral (46)
(49)	$\int \frac{x\,dx}{a^2 - x^2} = -\frac{1}{2} \cdot \ln \lvert a^2 - x^2 \rvert$
(50)	$\int \frac{x\,dx}{(a^2 - x^2)^2} = \frac{1}{2(a^2 - x^2)}$
(51)	$\int \frac{x\,dx}{(a^2 - x^2)^n} = \frac{1}{2(n-1)\,(a^2 - x^2)^{n-1}} \qquad (n \neq 1)$ Fall $n = 1$: siehe Integral (49)
(52)	$\int \frac{x^2\,dx}{a^2 - x^2} = -x + \frac{a}{2} \cdot \ln \left\lvert \frac{a+x}{a-x} \right\rvert$
(53)	$\int \frac{x^2\,dx}{(a^2 - x^2)^2} = \frac{x}{2(a^2 - x^2)} - \frac{1}{4a} \cdot \ln \left\lvert \frac{a+x}{a-x} \right\rvert$
(54)	$\int \frac{x^2\,dx}{(a^2 - x^2)^n} = \frac{x}{2(n-1)\,(a^2 - x^2)^{n-1}} - \frac{1}{2(n-1)} \cdot \underbrace{\int \frac{dx}{(a^2 - x^2)^{n-1}}}_{\text{Integral (48)}} \qquad (n \neq 1)$ Fall $n = 1$: siehe Integral (52)
(55)	$\int \frac{dx}{x(a^2 - x^2)} = -\frac{1}{2a^2} \cdot \ln \left\lvert \frac{a^2 - x^2}{x^2} \right\rvert$
(56)	$\int \frac{dx}{x(a^2 - x^2)^2} = \frac{1}{2a^2(a^2 - x^2)} - \frac{1}{2a^4} \cdot \ln \left\lvert \frac{a^2 - x^2}{x^2} \right\rvert$
(57)	$\int \frac{dx}{x^2(a^2 - x^2)} = -\frac{1}{a^2 x} + \frac{1}{2a^3} \cdot \ln \left\lvert \frac{a+x}{a-x} \right\rvert$
(58)	$\int \frac{dx}{x^2(a^2 - x^2)^2} = -\frac{1}{a^4 x} + \frac{x}{2a^4(a^2 - x^2)} + \frac{3}{4a^5} \cdot \ln \left\lvert \frac{a+x}{a-x} \right\rvert$
(59)	$\int \frac{x^m\,dx}{(a^2 - x^2)^n} = a^2 \cdot \int \frac{x^{m-2}\,dx}{(a^2 - x^2)^n} - \int \frac{x^{m-2}\,dx}{(a^2 - x^2)^{n-1}}$
(60)	$\int \frac{dx}{x^m(a^2 - x^2)^n} = \frac{1}{a^2} \cdot \int \frac{dx}{x^m(a^2 - x^2)^{n-1}} + \frac{1}{a^2} \cdot \int \frac{dx}{x^{m-2}(a^2 - x^2)^n}$
(61)	$\int \frac{dx}{(px+q)(a^2 - x^2)} = \frac{1}{a^2 p^2 - q^2} \left[\frac{p}{2} \cdot \ln \left\lvert \frac{(px+q)^2}{a^2 - x^2} \right\rvert - \frac{q}{2a} \cdot \ln \left\lvert \frac{a+x}{a-x} \right\rvert \right] \qquad (p \neq 0)$
(62)	$\int \frac{x\,dx}{(px+q)(a^2 - x^2)} = \frac{1}{2(a^2 p^2 - q^2)} \left[q \cdot \ln \left\lvert \frac{a^2 - x^2}{(px+q)^2} \right\rvert + ap \cdot \ln \left\lvert \frac{a+x}{a-x} \right\rvert \right] \qquad (p \neq 0)$

5 Integrale mit $ax^2 + bx + c \quad (a \neq 0)$

Abkürzung: $\boxed{\Delta = 4ac - b^2}$

Hinweis: Es wird stets $\Delta \neq 0$ vorausgesetzt. Für $\Delta = 0$ ist $ax^2 + bx + c = a\left(x + \frac{b}{2a}\right)^2$. Die Integrale entsprechen dann dem Integraltyp aus Abschnitt 1.

(63) $$\int \frac{dx}{ax^2 + bx + c} = \begin{cases} \dfrac{2}{\sqrt{\Delta}} \cdot \arctan\left(\dfrac{2ax + b}{\sqrt{\Delta}}\right) & \text{für } \Delta > 0 \\[2ex] \dfrac{1}{\sqrt{|\Delta|}} \cdot \ln\left|\dfrac{2ax + b - \sqrt{|\Delta|}}{2ax + b + \sqrt{|\Delta|}}\right| & \text{für } \Delta < 0 \end{cases}$$

(64) $$\int \frac{dx}{(ax^2 + bx + c)^2} = \frac{2ax + b}{\Delta(ax^2 + bx + c)} + \frac{2a}{\Delta} \cdot \underbrace{\int \frac{dx}{ax^2 + bx + c}}_{\text{Integral (63)}}$$

(65) $$\int \frac{dx}{(ax^2 + bx + c)^n} = \frac{2ax + b}{(n-1)\Delta(ax^2 + bx + c)^{n-1}} + \frac{2(2n-3)a}{(n-1)\Delta} \cdot \int \frac{dx}{(ax^2 + bx + c)^{n-1}} \quad (n \neq 1)$$

Fall $n = 1$: siehe Integral (63)

(66) $$\int \frac{x\,dx}{ax^2 + bx + c} = \frac{1}{2a} \cdot \ln|ax^2 + bx + c| - \frac{b}{2a} \cdot \underbrace{\int \frac{dx}{ax^2 + bx + c}}_{\text{Integral (63)}}$$

(67) $$\int \frac{x\,dx}{(ax^2 + bx + c)^2} = -\frac{bx + 2c}{\Delta(ax^2 + bx + c)} - \frac{b}{\Delta} \cdot \underbrace{\int \frac{dx}{ax^2 + bx + c}}_{\text{Integral (63)}}$$

(68) $$\int \frac{x\,dx}{(ax^2 + bx + c)^n} = -\frac{bx + 2c}{(n-1)\Delta(ax^2 + bx + c)^{n-1}} - \frac{(2n-3)b}{(n-1)\Delta} \cdot \underbrace{\int \frac{dx}{(ax^2 + bx + c)^{n-1}}}_{\text{Integral (65)}} \quad (n \neq 1)$$

Fall $n = 1$: siehe Integral (66)

(69) $$\int \frac{px + q}{ax^2 + bx + c}\,dx = \frac{p}{2a} \cdot \ln|ax^2 + bx + c| + \frac{2aq - bp}{2a} \cdot \underbrace{\int \frac{dx}{ax^2 + bx + c}}_{\text{Integral (63)}}$$

(70) $$\int \frac{px + q}{(ax^2 + bx + c)^n}\,dx = \frac{(2aq - bp)x + bq - 2cp}{(n-1)\Delta(ax^2 + bx + c)^{n-1}} + \frac{(2n-3)(2aq - bp)}{(n-1)\Delta} \cdot \underbrace{\int \frac{dx}{(ax^2 + bx + c)^{n-1}}}_{\text{Integral (65)}} \quad (n \neq 1)$$

Fall $n = 1$: siehe Integral (69)

(71) $$\int \frac{x^2\,dx}{ax^2 + bx + c} = \frac{x}{a} - \frac{b}{2a^2} \cdot \ln|ax^2 + bx + c| + \frac{b^2 - 2ac}{2a^2} \cdot \underbrace{\int \frac{dx}{ax^2 + bx + c}}_{\text{Integral (63)}}$$

(72) $$\int \frac{x^2\,dx}{(ax^2 + bx + c)^n} = -\frac{x}{(2n - 3)\,a\,(ax^2 + bx + c)^{n-1}} + \frac{c}{(2n - 3)\,a} \cdot \underbrace{\int \frac{dx}{(ax^2 + bx + c)^n}}_{\text{Integral (65)}} - \frac{(n - 2)\,b}{(2n - 3)\,a} \cdot \underbrace{\int \frac{x\,dx}{(ax^2 + bx + c)^n}}_{\text{Integral (68)}}$$

(73) $$\int \frac{dx}{x\,(ax^2 + bx + c)} = -\frac{1}{2c} \cdot \ln\left|\frac{ax^2 + bx + c}{x^2}\right| - \frac{b}{2c} \cdot \underbrace{\int \frac{dx}{ax^2 + bx + c}}_{\text{Integral (63)}} \qquad (c \neq 0)$$

(74) $$\int \frac{dx}{x\,(ax^2 + bx + c)^n} = \frac{1}{2\,(n - 1)\,c\,(ax^2 + bx + c)^{n-1}} - \frac{b}{2c} \cdot \underbrace{\int \frac{dx}{(ax^2 + bx + c)^n}}_{\text{Integral (65)}} + \frac{1}{c} \cdot \int \frac{dx}{x\,(ax^2 + bx + c)^{n-1}} \qquad (n \neq 1;\ c \neq 0)$$

Fall $n = 1$: siehe Integral (73)

(75) $$\int \frac{x^m\,dx}{(ax^2 + bx + c)^n} = -\frac{x^{m-1}}{(2n - m - 1)\,a\,(ax^2 + bx + c)^{n-1}} + \frac{(m - 1)\,c}{(2n - m - 1)\,a} \cdot \int \frac{x^{m-2}\,dx}{(ax^2 + bx + c)^n} + \frac{(m - n)\,b}{(2n - m - 1)\,a} \cdot \int \frac{x^{m-1}\,dx}{(ax^2 + bx + c)^n} \qquad (m \neq 2n - 1)$$

Fall $m = 2n - 1$: siehe Integral (76)

(76) $$\int \frac{x^{2n-1}\,dx}{(ax^2 + bx + c)^n} = \frac{1}{a} \cdot \int \frac{x^{2n-3}\,dx}{(ax^2 + bx + c)^{n-1}} - \frac{c}{a} \cdot \int \frac{x^{2n-3}\,dx}{(ax^2 + bx + c)^n} - \frac{b}{a} \cdot \int \frac{x^{2n-2}\,dx}{(ax^2 + bx + c)^n}$$

(77) $$\int \frac{dx}{x^m\,(ax^2 + bx + c)^n} = -\frac{1}{(m - 1)\,c\,x^{m-1}\,(ax^2 + bx + c)^{n-1}} - \frac{(m + 2n - 3)\,a}{(m - 1)\,c} \cdot \int \frac{dx}{x^{m-2}\,(ax^2 + bx + c)^n} - \frac{(m + n - 2)\,b}{(m - 1)\,c} \cdot \int \frac{dx}{x^{m-1}\,(ax^2 + bx + c)^n}$$

$(m \neq 1;\ c \neq 0)$ Fall $m = 1$: siehe Integral (74)

(78) $$\int \frac{dx}{(px+q)(ax^2+bx+c)} = \frac{1}{2(aq^2-bpq+cp^2)} \left[p \cdot \ln \left| \frac{(px+q)^2}{ax^2+bx+c} \right| + (2aq-bp) \cdot \underbrace{\int \frac{dx}{ax^2+bx+c}}_{\text{Integral (63)}} \right] \quad (p \neq 0)$$

6 Integrale mit $a^3 \pm x^3 \quad (a > 0)$

Hinweis: Das *obere* Vorzeichen gilt für $a^3 + x^3$, das *untere* Vorzeichen für $a^3 - x^3$.

(79) $$\int \frac{dx}{a^3 \pm x^3} = \pm \frac{1}{6a^2} \cdot \ln \left| \frac{(a \pm x)^2}{a^2 \mp ax + x^2} \right| + \frac{1}{a^2\sqrt{3}} \cdot \arctan \left(\frac{2x \mp a}{a\sqrt{3}} \right)$$

(80) $$\int \frac{dx}{(a^3 \pm x^3)^2} = \frac{x}{3a^3(a^3 \pm x^3)} \pm \frac{1}{9a^5} \cdot \ln \left| \frac{(a \pm x)^2}{a^2 \mp ax + x^2} \right| + \frac{2}{3a^5\sqrt{3}} \cdot \arctan \left(\frac{2x \mp a}{a\sqrt{3}} \right)$$

(81) $$\int \frac{x\,dx}{a^3 \pm x^3} = \frac{1}{6a} \cdot \ln \left| \frac{a^2 \mp ax + x^2}{(a \pm x)^2} \right| \pm \frac{1}{a^2\sqrt{3}} \cdot \arctan \left(\frac{2x \mp a}{a\sqrt{3}} \right)$$

(82) $$\int \frac{x\,dx}{(a^3 \pm x^3)^2} = \frac{x^2}{3a^3(a^3 \pm x^3)} + \frac{1}{18a^4} \cdot \ln \left| \frac{a^2 \mp ax + x^2}{(a \pm x)^2} \right| \pm \frac{1}{3a^4\sqrt{3}} \cdot \arctan \left(\frac{2x \mp a}{a\sqrt{3}} \right)$$

(83) $$\int \frac{dx}{x(a^3 \pm x^3)} = \frac{1}{3a^3} \cdot \ln \left| \frac{x^3}{a^3 \pm x^3} \right|$$

7 Integrale mit $a^4 + x^4 \quad (a > 0)$

(84) $$\int \frac{dx}{a^4 + x^4} = \frac{1}{4\sqrt{2}a^3} \cdot \ln \left| \frac{x^2 + a\sqrt{2}x + a^2}{x^2 - a\sqrt{2}x + a^2} \right| - \frac{1}{2\sqrt{2}a^3} \cdot \arctan \left(\frac{a\sqrt{2}x}{x^2 - a^2} \right)$$

(85) $$\int \frac{x\,dx}{a^4 + x^4} = \frac{1}{2a^2} \cdot \arctan \left(\frac{x^2}{a^2} \right)$$

(86) $$\int \frac{dx}{x(a^4 + x^4)} = \frac{1}{4a^4} \cdot \ln \left(\frac{x^4}{a^4 + x^4} \right)$$

8 Integrale mit $a^4 - x^4 \quad (a > 0)$

(87) $$\int \frac{dx}{a^4 - x^4} = \frac{1}{4a^3} \cdot \ln \left| \frac{a + x}{a - x} \right| + \frac{1}{2a^3} \cdot \arctan \left(\frac{x}{a} \right)$$

(88)	$\int \frac{x\,dx}{a^4 - x^4} = \frac{1}{4a^2} \cdot \ln \left\| \frac{a^2 + x^2}{a^2 - x^2} \right\|$
(89)	$\int \frac{dx}{x(a^4 - x^4)} = -\frac{1}{4a^4} \cdot \ln \left\| \frac{a^4 - x^4}{x^4} \right\|$

9 Integrale mit $\sqrt{ax + b}$ $(a \neq 0)$

(90)	$\int \sqrt{ax + b}\,dx = \frac{2}{3a}\sqrt{(ax + b)^3}$
(91)	$\int x\sqrt{ax + b}\,dx = \frac{2(3ax - 2b)}{15a^2}\sqrt{(ax + b)^3}$
(92)	$\int x^n\sqrt{ax + b}\,dx = \frac{2x^n}{(2n + 3)a}\sqrt{(ax + b)^3} - \frac{2nb}{(2n + 3)a} \cdot \int x^{n-1}\sqrt{ax + b}\,dx$
(93)	$\int \frac{\sqrt{ax + b}}{x}\,dx = 2\sqrt{ax + b} + b \cdot \underbrace{\int \frac{dx}{x\sqrt{ax + b}}}_{\text{Integral (99)}}$
(94)	$\int \frac{\sqrt{ax + b}}{x^2}\,dx = -\frac{\sqrt{ax + b}}{x} + \frac{a}{2} \cdot \underbrace{\int \frac{dx}{x\sqrt{ax + b}}}_{\text{Integral (99)}}$
(95)	$\int \frac{\sqrt{ax + b}}{x^n}\,dx = -\frac{\sqrt{(ax + b)^3}}{(n - 1)bx^{n-1}} - \frac{(2n - 5)a}{2(n - 1)b} \cdot \int \frac{\sqrt{ax + b}}{x^{n-1}}\,dx \qquad (n \neq 1;\ b \neq 0)$ Fall $n = 1$: siehe Integral (93)
(96)	$\int \frac{dx}{\sqrt{ax + b}} = \frac{2}{a} \cdot \sqrt{ax + b}$
(97)	$\int \frac{x\,dx}{\sqrt{ax + b}} = \frac{2(ax - 2b)}{3a^2} \cdot \sqrt{ax + b}$
(98)	$\int \frac{x^n\,dx}{\sqrt{ax + b}} = \frac{2x^n\sqrt{ax + b}}{(2n + 1)a} - \frac{2nb}{(2n + 1)a} \cdot \int \frac{x^{n-1}\,dx}{\sqrt{ax + b}}$
(99)	$\int \frac{dx}{x\sqrt{ax + b}} = \begin{cases} \frac{1}{\sqrt{b}} \cdot \ln \left\| \frac{\sqrt{ax + b} - \sqrt{b}}{\sqrt{ax + b} + \sqrt{b}} \right\| & \text{für} \quad b > 0 \\ \frac{2}{\sqrt{\|b\|}} \cdot \arctan\left(\sqrt{\frac{ax + b}{\|b\|}}\right) & \text{für} \quad b < 0 \end{cases}$

(100)	$\displaystyle\int \frac{dx}{x^n\sqrt{ax+b}} = -\frac{\sqrt{ax+b}}{(n-1)\,b\,x^{n-1}} - \frac{(2n-3)\,a}{2\,(n-1)\,b} \cdot \int \frac{dx}{x^{n-1}\sqrt{ax+b}} \qquad (n \neq 1;\ b \neq 0)$ Fall $n = 1$: siehe Integral (99)
(101)	$\displaystyle\int \sqrt{(ax+b)^3}\,dx = \frac{2}{5a} \cdot \sqrt{(ax+b)^5}$
(102)	$\displaystyle\int \sqrt{(ax+b)^n}\,dx = \frac{2\sqrt{(ax+b)^{n+2}}}{(n+2)\,a} \qquad (n \neq -2)$ Fall $n = -2$: siehe Integral (2)
(103)	$\displaystyle\int x\sqrt{(ax+b)^3}\,dx = \frac{2}{35\,a^2}\left[5\sqrt{(ax+b)^7} - 7b\sqrt{(ax+b)^5}\right]$
(104)	$\displaystyle\int x\sqrt{(ax+b)^n}\,dx = \frac{2\sqrt{(ax+b)^{n+4}}}{(n+4)\,a^2} - \frac{2b\sqrt{(ax+b)^{n+2}}}{(n+2)\,a^2} \qquad (n \neq -2, -4)$ Fall $n = -2, -4$: siehe Integral (4) bzw. (5)
(105)	$\displaystyle\int \sqrt{\frac{(ax+b)^3}{x}}\,dx = \frac{2}{3}\sqrt{(ax+b)^3} + 2b\sqrt{ax+b} + b^2 \cdot \underbrace{\int \frac{dx}{x\sqrt{ax+b}}}_{\text{Integral (99)}}$
(106)	$\displaystyle\int \frac{x}{\sqrt{(ax+b)^3}}\,dx = \frac{2}{a^2}\left[\sqrt{ax+b} + \frac{b}{\sqrt{ax+b}}\right] = \frac{2\,(ax+2b)}{a^2\sqrt{ax+b}}$

10 Integrale mit $\sqrt{ax+b}$ und $px+q$

Abkürzung: $\boxed{\Delta = bp - aq}$

Hinweis: Es wird stets $\Delta \neq 0$ vorausgesetzt. Für $\Delta = 0$ ist $px + q = \frac{q}{b}\,(ax+b)$.
Die Integrale entsprechen dann dem Integraltyp aus Abschnitt 9.

(107)	$\displaystyle\int \frac{\sqrt{ax+b}}{px+q}\,dx = \begin{cases} \dfrac{2\sqrt{ax+b}}{p} + \dfrac{\sqrt{\Delta}}{p\sqrt{p}} \cdot \ln\left\lvert\dfrac{\sqrt{p\,(ax+b)} - \sqrt{\Delta}}{\sqrt{p\,(ax+b)} + \sqrt{\Delta}}\right\rvert & \text{für } p > 0,\ \Delta > 0 \\[2ex] \dfrac{2\sqrt{ax+b}}{p} - \dfrac{2\sqrt{\lvert\Delta\rvert}}{p\sqrt{p}} \cdot \arctan\left(\sqrt{\dfrac{p\,(ax+b)}{\lvert\Delta\rvert}}\right) & \text{für } p > 0,\ \Delta < 0 \end{cases}$
(108)	$\displaystyle\int \frac{\sqrt{ax+b}}{(px+q)^n}\,dx = -\frac{\sqrt{ax+b}}{(n-1)\,p\,(px+q)^{n-1}} + \frac{a}{2\,(n-1)\,p} \cdot \underbrace{\int \frac{dx}{(px+q)^{n-1}\sqrt{ax+b}}}_{\text{Integral (111)}} \qquad (n \neq 1)$ Fall $n = 1$: siehe Integral (107)

(109) $$\int \frac{px+q}{\sqrt{ax+b}}\,dx = \frac{2(apx+3aq-2bp)\sqrt{ax+b}}{3a^2}$$

(110) $$\int \frac{dx}{(px+q)\sqrt{ax+b}} = \begin{cases} \dfrac{1}{\sqrt{p\Delta}} \cdot \ln\left|\dfrac{\sqrt{p(ax+b)}-\sqrt{\Delta}}{\sqrt{p(ax+b)}+\sqrt{\Delta}}\right| & \text{für } \Delta > 0,\ p > 0 \\ \dfrac{2}{\sqrt{p|\Delta|}} \cdot \arctan\left(\sqrt{\dfrac{p(ax+b)}{|\Delta|}}\right) & \text{für } \Delta < 0,\ p > 0 \end{cases}$$

(111) $$\int \frac{dx}{(px+q)^n\sqrt{ax+b}} = -\frac{\sqrt{ax+b}}{(n-1)\Delta(px+q)^{n-1}} - \frac{(2n-3)a}{2(n-1)\Delta} \cdot \int \frac{dx}{(px+q)^{n-1}\sqrt{ax+b}} \qquad (n \neq 1)$$

Fall $n = 1$: siehe Integral (110)

11 Integrale mit $\sqrt{ax+b}$ und $\sqrt{px+q}$ $(a, p \neq 0)$

Abkürzung: $\boxed{\Delta = bp - aq}$

(112) $$\int \sqrt{(ax+b)(px+q)}\,dx = \frac{[2a(px+q)+\Delta]\sqrt{(ax+b)(px+q)}}{4ap} - \frac{\Delta^2}{8ap} \cdot \underbrace{\int \frac{dx}{\sqrt{(ax+b)(px+q)}}}_{\text{Integral (114)}}$$

(113) $$\int \sqrt{\frac{px+q}{ax+b}}\,dx = \frac{\sqrt{(ax+b)(px+q)}}{a} - \frac{\Delta}{2a} \cdot \underbrace{\int \frac{dx}{\sqrt{(ax+b)(px+q)}}}_{\text{Integral (114)}}$$

(114) $$\int \frac{dx}{\sqrt{(ax+b)(px+q)}} = \begin{cases} \dfrac{2}{\sqrt{ap}} \cdot \ln\left|\sqrt{a(px+q)} + \sqrt{p(ax+b)}\right| & \text{für } ap > 0 \\ -\dfrac{2}{\sqrt{|ap|}} \cdot \arctan\left(\sqrt{-\dfrac{p(ax+b)}{a(px+q)}}\right) & \text{für } ap < 0 \end{cases}$$

(115) $$\int \frac{x\,dx}{\sqrt{(ax+b)(px+q)}} = \frac{\sqrt{(ax+b)(px+q)}}{ap} - \frac{aq+bp}{2ap} \cdot \underbrace{\int \frac{dx}{\sqrt{(ax+b)(px+q)}}}_{\text{Integral (114)}}$$

12 Integrale mit $\sqrt{a^2+x^2}$ $(a>0)$

(116)	$\int \sqrt{a^2+x^2}\,dx = \frac{1}{2}\left[x\sqrt{a^2+x^2} + a^2 \cdot \ln\left(x+\sqrt{a^2+x^2}\right)\right] = \frac{1}{2}\left[x\sqrt{a^2+x^2} + a^2 \cdot \operatorname{arsinh}\left(\frac{x}{a}\right)\right]$
(117)	$\int x\sqrt{a^2+x^2}\,dx = \frac{1}{3}\sqrt{(a^2+x^2)^3}$
(118)	$\int x^2\sqrt{a^2+x^2}\,dx = \frac{1}{4}x\sqrt{(a^2+x^2)^3} - \frac{a^2}{8}\left[x\sqrt{a^2+x^2} + a^2 \cdot \ln\left(x+\sqrt{a^2+x^2}\right)\right] = \frac{1}{4}x\sqrt{(a^2+x^2)^3} - \frac{a^2}{8}\left[x\sqrt{a^2+x^2} + a^2 \cdot \operatorname{arsinh}\left(\frac{x}{a}\right)\right]$
(119)	$\int x^3\sqrt{a^2+x^2}\,dx = \frac{1}{5}\sqrt{(a^2+x^2)^5} - \frac{a^2}{3}\sqrt{(a^2+x^2)^3}$
(120)	$\int \frac{\sqrt{a^2+x^2}}{x}\,dx = \sqrt{a^2+x^2} - a \cdot \ln\left\lvert\frac{a+\sqrt{a^2+x^2}}{x}\right\rvert$
(121)	$\int \frac{\sqrt{a^2+x^2}}{x^2}\,dx = -\frac{\sqrt{a^2+x^2}}{x} + \ln\left(x+\sqrt{a^2+x^2}\right) = -\frac{\sqrt{a^2+x^2}}{x} + \operatorname{arsinh}\left(\frac{x}{a}\right)$
(122)	$\int \frac{\sqrt{a^2+x^2}}{x^3}\,dx = -\frac{\sqrt{a^2+x^2}}{2x^2} - \frac{1}{2a} \cdot \ln\left\lvert\frac{a+\sqrt{a^2+x^2}}{x}\right\rvert$
(123)	$\int \frac{dx}{\sqrt{a^2+x^2}} = \ln\left(x+\sqrt{a^2+x^2}\right) = \operatorname{arsinh}\left(\frac{x}{a}\right)$
(124)	$\int \frac{x\,dx}{\sqrt{a^2+x^2}} = \sqrt{a^2+x^2}$
(125)	$\int \frac{x^2\,dx}{\sqrt{a^2+x^2}} = \frac{1}{2}x\sqrt{a^2+x^2} - \frac{a^2}{2} \cdot \ln\left(x+\sqrt{a^2+x^2}\right) = \frac{1}{2}x\sqrt{a^2+x^2} - \frac{a^2}{2} \cdot \operatorname{arsinh}\left(\frac{x}{a}\right)$
(126)	$\int \frac{x^3\,dx}{\sqrt{a^2+x^2}} = \frac{1}{3}\sqrt{(a^2+x^2)^3} - a^2\sqrt{a^2+x^2}$
(127)	$\int \frac{dx}{x\sqrt{a^2+x^2}} = -\frac{1}{a} \cdot \ln\left\lvert\frac{a+\sqrt{a^2+x^2}}{x}\right\rvert$

(128) $$\int \frac{dx}{x^2 \sqrt{a^2 + x^2}} = -\frac{\sqrt{a^2 + x^2}}{a^2 x}$$

(129) $$\int \frac{dx}{x^3 \sqrt{a^2 + x^2}} = -\frac{\sqrt{a^2 + x^2}}{2a^2 x^2} + \frac{1}{2a^3} \cdot \ln \left| \frac{a + \sqrt{a^2 + x^2}}{x} \right|$$

(130) $$\int \sqrt{(a^2 + x^2)^3}\, dx = \frac{1}{4} \left[x \sqrt{(a^2 + x^2)^3} + \frac{3}{2} a^2 x \sqrt{a^2 + x^2} + \frac{3}{2} a^4 \cdot \ln \left(x + \sqrt{a^2 + x^2} \right) \right] =$$
$$= \frac{1}{4} \left[x \sqrt{(a^2 + x^2)^3} + \frac{3}{2} a^2 x \sqrt{a^2 + x^2} + \frac{3}{2} a^4 \cdot \operatorname{arsinh} \left(\frac{x}{a} \right) \right]$$

(131) $$\int x \sqrt{(a^2 + x^2)^3}\, dx = \frac{1}{5} \sqrt{(a^2 + x^2)^5}$$

(132) $$\int x^2 \sqrt{(a^2 + x^2)^3}\, dx = \frac{1}{6} x \sqrt{(a^2 + x^2)^5} - \frac{a^2}{24} x \sqrt{(a^2 + x^2)^3} - \frac{a^4}{16} x \sqrt{a^2 + x^2} -$$
$$- \frac{a^6}{16} \cdot \ln \left(x + \sqrt{a^2 + x^2} \right) =$$
$$= \frac{1}{6} x \sqrt{(a^2 + x^2)^5} - \frac{a^2}{24} x \sqrt{(a^2 + x^2)^3} - \frac{a^4}{16} x \sqrt{a^2 + x^2} -$$
$$- \frac{a^6}{16} \cdot \operatorname{arsinh} \left(\frac{x}{a} \right)$$

(133) $$\int \frac{\sqrt{(a^2 + x^2)^3}}{x}\, dx = \frac{1}{3} \sqrt{(a^2 + x^2)^3} + a^2 \sqrt{a^2 + x^2} - a^3 \cdot \ln \left| \frac{a + \sqrt{a^2 + x^2}}{x} \right|$$

(134) $$\int \frac{\sqrt{(a^2 + x^2)^3}}{x^2}\, dx = -\frac{\sqrt{(a^2 + x^2)^3}}{x} + \frac{3}{2} x \sqrt{a^2 + x^2} + \frac{3}{2} a^2 \cdot \ln \left(x + \sqrt{a^2 + x^2} \right) =$$
$$= -\frac{\sqrt{(a^2 + x^2)^3}}{x} + \frac{3}{2} x \sqrt{a^2 + x^2} + \frac{3}{2} a^2 \cdot \operatorname{arsinh} \left(\frac{x}{a} \right)$$

(135) $$\int \frac{dx}{\sqrt{(a^2 + x^2)^3}} = \frac{x}{a^2 \sqrt{a^2 + x^2}}$$

(136) $$\int \frac{x\, dx}{\sqrt{(a^2 + x^2)^3}} = -\frac{1}{\sqrt{a^2 + x^2}}$$

(137) $$\int \frac{x^2\, dx}{\sqrt{(a^2 + x^2)^3}} = -\frac{x}{\sqrt{a^2 + x^2}} + \ln \left(x + \sqrt{a^2 + x^2} \right) = -\frac{x}{\sqrt{a^2 + x^2}} + \operatorname{arsinh} \left(\frac{x}{a} \right)$$

(138)	$\int \frac{dx}{x\sqrt{(a^2+x^2)^3}} = \frac{1}{a^2\sqrt{a^2+x^2}} - \frac{1}{a^3} \cdot \ln\left\lvert \frac{a+\sqrt{a^2+x^2}}{x} \right\rvert$
(139)	$\int \frac{dx}{x^2\sqrt{(a^2+x^2)^3}} = -\frac{2x^2+a^2}{a^4 x\sqrt{a^2+x^2}}$
(140)	$\int \frac{dx}{(px+q)\sqrt{a^2+x^2}} = -\frac{1}{\sqrt{a^2p^2+q^2}} \cdot \ln\left\lvert \frac{\sqrt{a^2p^2+q^2}\cdot\sqrt{a^2+x^2} - qx + a^2p}{px+q} \right\rvert$ $(p \neq 0)$

13 Integrale mit $\sqrt{a^2 - x^2}$ $(a > 0;\ \lvert x \rvert < a)$

(141)	$\int \sqrt{a^2-x^2}\,dx = \frac{1}{2}\left[x\sqrt{a^2-x^2} + a^2 \cdot \arcsin\left(\frac{x}{a}\right)\right]$
(142)	$\int x\sqrt{a^2-x^2}\,dx = -\frac{1}{3}\sqrt{(a^2-x^2)^3}$
(143)	$\int x^2\sqrt{a^2-x^2}\,dx = -\frac{1}{4}x\sqrt{(a^2-x^2)^3} + \frac{a^2}{8}\left[x\sqrt{a^2-x^2} + a^2 \cdot \arcsin\left(\frac{x}{a}\right)\right]$
(144)	$\int x^3\sqrt{a^2-x^2}\,dx = \frac{1}{5}\sqrt{(a^2-x^2)^5} - \frac{a^2}{3}\sqrt{(a^2-x^2)^3}$
(145)	$\int \frac{\sqrt{a^2-x^2}}{x}\,dx = \sqrt{a^2-x^2} - a \cdot \ln\left\lvert \frac{a+\sqrt{a^2-x^2}}{x} \right\rvert$
(146)	$\int \frac{\sqrt{a^2-x^2}}{x^2}\,dx = -\frac{\sqrt{a^2-x^2}}{x} - \arcsin\left(\frac{x}{a}\right)$
(147)	$\int \frac{\sqrt{a^2-x^2}}{x^3}\,dx = -\frac{\sqrt{a^2-x^2}}{2x^2} + \frac{1}{2a} \cdot \ln\left\lvert \frac{a+\sqrt{a^2-x^2}}{x} \right\rvert$
(148)	$\int \frac{dx}{\sqrt{a^2-x^2}} = \arcsin\left(\frac{x}{a}\right)$
(149)	$\int \frac{x\,dx}{\sqrt{a^2-x^2}} = -\sqrt{a^2-x^2}$
(150)	$\int \frac{x^2\,dx}{\sqrt{a^2-x^2}} = -\frac{1}{2}x\sqrt{a^2-x^2} + \frac{a^2}{2} \cdot \arcsin\left(\frac{x}{a}\right)$

(151) $$\int \frac{x^3\,dx}{\sqrt{a^2-x^2}} = \frac{1}{3}\sqrt{(a^2-x^2)^3} - a^2\sqrt{a^2-x^2}$$

(152) $$\int \frac{dx}{x\sqrt{a^2-x^2}} = -\frac{1}{a}\cdot\ln\left|\frac{a+\sqrt{a^2-x^2}}{x}\right|$$

(153) $$\int \frac{dx}{x^2\sqrt{a^2-x^2}} = -\frac{\sqrt{a^2-x^2}}{a^2x}$$

(154) $$\int \frac{dx}{x^3\sqrt{a^2-x^2}} = -\frac{\sqrt{a^2-x^2}}{2a^2x^2} - \frac{1}{2a^3}\cdot\ln\left|\frac{a+\sqrt{a^2-x^2}}{x}\right|$$

(155) $$\int \sqrt{(a^2-x^2)^3}\,dx = \frac{1}{4}\left[x\sqrt{(a^2-x^2)^3} + \frac{3}{2}a^2x\sqrt{a^2-x^2} + \frac{3}{2}a^4\cdot\arcsin\left(\frac{x}{a}\right)\right]$$

(156) $$\int x\sqrt{(a^2-x^2)^3}\,dx = -\frac{1}{5}\sqrt{(a^2-x^2)^5}$$

(157) $$\int x^2\sqrt{(a^2-x^2)^3}\,dx = -\frac{1}{6}x\sqrt{(a^2-x^2)^5} + \frac{a^2}{24}x\sqrt{(a^2-x^2)^3} + \frac{a^4}{16}x\sqrt{a^2-x^2} + {}$$
$$+ \frac{a^6}{16}\cdot\arcsin\left(\frac{x}{a}\right)$$

(158) $$\int \frac{\sqrt{(a^2-x^2)^3}}{x}\,dx = \frac{1}{3}\sqrt{(a^2-x^2)^3} + a^2\sqrt{a^2-x^2} - a^3\cdot\ln\left|\frac{a+\sqrt{a^2-x^2}}{x}\right|$$

(159) $$\int \frac{\sqrt{(a^2-x^2)^3}}{x^2}\,dx = -\frac{\sqrt{(a^2-x^2)^3}}{x} - \frac{3}{2}x\sqrt{a^2-x^2} - \frac{3}{2}a^2\cdot\arcsin\left(\frac{x}{a}\right)$$

(160) $$\int \frac{dx}{\sqrt{(a^2-x^2)^3}} = \frac{x}{a^2\sqrt{a^2-x^2}}$$

(161) $$\int \frac{x\,dx}{\sqrt{(a^2-x^2)^3}} = \frac{1}{\sqrt{a^2-x^2}}$$

(162) $$\int \frac{x^2\,dx}{\sqrt{(a^2-x^2)^3}} = \frac{x}{\sqrt{a^2-x^2}} - \arcsin\left(\frac{x}{a}\right)$$

(163) $$\int \frac{dx}{x\sqrt{(a^2-x^2)^3}} = \frac{1}{a^2\sqrt{a^2-x^2}} - \frac{1}{a^3}\cdot\ln\left|\frac{a+\sqrt{a^2-x^2}}{x}\right|$$

(164) $$\int \frac{dx}{x^2 \sqrt{(a^2 - x^2)^3}} = \frac{2x^2 - a^2}{a^4 x \sqrt{a^2 - x^2}}$$

14 Integrale mit $\sqrt{x^2 - a^2}$ $(a > 0;\ |x| > a)$

(165) $$\int \sqrt{x^2 - a^2}\, dx = \frac{1}{2} \left[x \sqrt{x^2 - a^2} - a^2 \cdot \ln \left| x + \sqrt{x^2 - a^2} \right| \right]$$

(166) $$\int x \sqrt{x^2 - a^2}\, dx = \frac{1}{3} \sqrt{(x^2 - a^2)^3}$$

(167) $$\int x^2 \sqrt{x^2 - a^2}\, dx = \frac{1}{4}\, x \sqrt{(x^2 - a^2)^3} + \frac{a^2}{8} \left[x \sqrt{x^2 - a^2} - a^2 \cdot \ln \left| x + \sqrt{x^2 - a^2} \right| \right]$$

(168) $$\int x^3 \sqrt{x^2 - a^2}\, dx = \frac{1}{5} \sqrt{(x^2 - a^2)^5} + \frac{a^2}{3} \sqrt{(x^2 - a^2)^3}$$

(169) $$\int \frac{\sqrt{x^2 - a^2}}{x}\, dx = \sqrt{x^2 - a^2} - a \cdot \arccos \left| \frac{a}{x} \right|$$

(170) $$\int \frac{\sqrt{x^2 - a^2}}{x^2}\, dx = -\frac{\sqrt{x^2 - a^2}}{x} + \ln \left| x + \sqrt{x^2 - a^2} \right|$$

(171) $$\int \frac{\sqrt{x^2 - a^2}}{x^3}\, dx = -\frac{\sqrt{x^2 - a^2}}{2x^2} + \frac{1}{2a} \cdot \arccos \left| \frac{a}{x} \right|$$

(172) $$\int \frac{dx}{\sqrt{x^2 - a^2}} = \ln \left| x + \sqrt{x^2 - a^2} \right| = \operatorname{sgn}(x) \cdot \operatorname{arcosh} \left| \frac{x}{a} \right|$$

(173) $$\int \frac{x\, dx}{\sqrt{x^2 - a^2}} = \sqrt{x^2 - a^2}$$

(174) $$\int \frac{x^2\, dx}{\sqrt{x^2 - a^2}} = \frac{1}{2}\, x \sqrt{x^2 - a^2} + \frac{a^2}{2} \cdot \ln \left| x + \sqrt{x^2 - a^2} \right|$$

(175) $$\int \frac{x^3\, dx}{\sqrt{x^2 - a^2}} = \frac{1}{3} \sqrt{(x^2 - a^2)^3} + a^2 \sqrt{x^2 - a^2}$$

(176) $$\int \frac{dx}{x \sqrt{x^2 - a^2}} = \frac{1}{a} \cdot \arccos \left| \frac{a}{x} \right|$$

(177)	$\displaystyle\int \frac{dx}{x^2\sqrt{x^2-a^2}} = \frac{\sqrt{x^2-a^2}}{a^2 x}$
(178)	$\displaystyle\int \frac{dx}{x^3\sqrt{x^2-a^2}} = \frac{\sqrt{x^2-a^2}}{2a^2x^2} + \frac{1}{2a^3}\cdot \arccos\left\|\frac{a}{x}\right\|$
(179)	$\displaystyle\int \sqrt{(x^2-a^2)^3}\,dx = \frac{1}{4}\left[x\sqrt{(x^2-a^2)^3} - \frac{3}{2}a^2x\sqrt{x^2-a^2} + \frac{3}{2}a^4\cdot\ln\left\|x+\sqrt{x^2-a^2}\right\|\right]$
(180)	$\displaystyle\int x\sqrt{(x^2-a^2)^3}\,dx = \frac{1}{5}\sqrt{(x^2-a^2)^5}$
(181)	$\displaystyle\int x^2\sqrt{(x^2-a^2)^3}\,dx = \frac{1}{6}x\sqrt{(x^2-a^2)^5} + \frac{a^2}{24}x\sqrt{(x^2-a^2)^3} - \frac{a^4}{16}x\sqrt{x^2-a^2} + {}$ $\displaystyle + \frac{a^6}{16}\cdot\ln\left\|x+\sqrt{x^2-a^2}\right\|$
(182)	$\displaystyle\int \frac{\sqrt{(x^2-a^2)^3}}{x}\,dx = \frac{1}{3}\sqrt{(x^2-a^2)^3} - a^2\sqrt{x^2-a^2} + a^3\cdot\arccos\left\|\frac{a}{x}\right\|$
(183)	$\displaystyle\int \frac{\sqrt{(x^2-a^2)^3}}{x^2}\,dx = -\frac{\sqrt{(x^2-a^2)^3}}{x} + \frac{3}{2}x\sqrt{x^2-a^2} - \frac{3}{2}a^2\cdot\ln\left\|x+\sqrt{x^2-a^2}\right\|$
(184)	$\displaystyle\int \frac{dx}{\sqrt{(x^2-a^2)^3}} = -\frac{x}{a^2\sqrt{x^2-a^2}}$
(185)	$\displaystyle\int \frac{x\,dx}{\sqrt{(x^2-a^2)^3}} = -\frac{1}{\sqrt{x^2-a^2}}$
(186)	$\displaystyle\int \frac{x^2\,dx}{\sqrt{(x^2-a^2)^3}} = -\frac{x}{\sqrt{x^2-a^2}} + \ln\left\|x+\sqrt{x^2-a^2}\right\|$
(187)	$\displaystyle\int \frac{dx}{x\sqrt{(x^2-a^2)^3}} = -\frac{1}{a^2\sqrt{x^2-a^2}} - \frac{1}{a^3}\cdot\arccos\left\|\frac{a}{x}\right\|$
(188)	$\displaystyle\int \frac{dx}{x^2\sqrt{(x^2-a^2)^3}} = \frac{a^2-2x^2}{a^4x\sqrt{x^2-a^2}}$

15 Integrale mit $\sqrt{a x^2 + b x + c}$ $(a \neq 0)$

Abkürzung: $\boxed{\Delta = 4 a c - b^2}$

Hinweis: Es wird stets $\Delta \neq 0$ vorausgesetzt. Für $\Delta = 0$ ist $\sqrt{a x^2 + b x + c} = \sqrt{a}\left(x + \dfrac{b}{2a}\right)$. Die Integrale entsprechen dann dem Integraltyp aus Abschnitt 1.

(189) $$\int \sqrt{a x^2 + b x + c}\, dx = \frac{2 a x + b}{4 a} \sqrt{a x^2 + b x + c} + \frac{\Delta}{8 a} \cdot \underbrace{\int \frac{dx}{\sqrt{a x^2 + b x + c}}}_{\text{Integral (194)}}$$

(190) $$\int x \sqrt{a x^2 + b x + c}\, dx = \frac{1}{3 a} \cdot \sqrt{(a x^2 + b x + c)^3} - \frac{b(2 a x + b)}{8 a^2} \sqrt{a x^2 + b x + c} - \frac{b \Delta}{16 a^2} \cdot \underbrace{\int \frac{dx}{\sqrt{a x^2 + b x + c}}}_{\text{Integral (194)}}$$

(191) $$\int x^2 \sqrt{a x^2 + b x + c}\, dx = \frac{1}{24 a^2} (6 a x - 5 b) \sqrt{(a x^2 + b x + c)^3} + \frac{5 b^2 - 4 a c}{16 a^2} \cdot \underbrace{\int \sqrt{a x^2 + b x + c}\, dx}_{\text{Integral (189)}}$$

(192) $$\int \frac{\sqrt{a x^2 + b x + c}}{x}\, dx = \sqrt{a x^2 + b x + c} + \frac{b}{2} \cdot \underbrace{\int \frac{dx}{\sqrt{a x^2 + b x + c}}}_{\text{Integral (194)}} + c \cdot \underbrace{\int \frac{dx}{x \sqrt{a x^2 + b x + c}}}_{\text{Integral (197)}}$$

(193) $$\int \frac{\sqrt{a x^2 + b x + c}}{x^2}\, dx = -\frac{\sqrt{a x^2 + b x + c}}{x} + a \cdot \underbrace{\int \frac{dx}{\sqrt{a x^2 + b x + c}}}_{\text{Integral (194)}} + \frac{b}{2} \cdot \underbrace{\int \frac{dx}{x \sqrt{a x^2 + b x + c}}}_{\text{Integral (197)}}$$

(194) $$\int \frac{dx}{\sqrt{a x^2 + b x + c}} = \begin{cases} \dfrac{1}{\sqrt{a}} \cdot \ln \left| 2 \sqrt{a} \sqrt{a x^2 + b x + c} + 2 a x + b \right| & \text{für } a > 0 \\[2ex] \dfrac{1}{\sqrt{a}} \cdot \operatorname{arsinh} \left(\dfrac{2 a x + b}{\sqrt{\Delta}} \right) & \text{für } a > 0,\ \Delta > 0 \\[2ex] -\dfrac{1}{\sqrt{|a|}} \cdot \arcsin \left(\dfrac{2 a x + b}{\sqrt{|\Delta|}} \right) & \text{für } a < 0,\ \Delta < 0 \end{cases}$$

(195) $$\int \frac{x\,dx}{\sqrt{ax^2+bx+c}} = \frac{\sqrt{ax^2+bx+c}}{a} - \frac{b}{2a}\cdot \underbrace{\int \frac{dx}{\sqrt{ax^2+bx+c}}}_{\text{Integral (194)}}$$

(196) $$\int \frac{x^2\,dx}{\sqrt{ax^2+bx+c}} = \frac{2ax-3b}{4a^2}\sqrt{ax^2+bx+c} + \frac{3b^2-4ac}{8a^2}\cdot \underbrace{\int \frac{dx}{\sqrt{ax^2+bx+c}}}_{\text{Integral (194)}}$$

(197) $$\int \frac{dx}{x\sqrt{ax^2+bx+c}} = \begin{cases} -\dfrac{1}{\sqrt{c}}\cdot \ln\left|\dfrac{2\sqrt{c}\sqrt{ax^2+bx+c}+bx+2c}{x}\right| & \text{für} \quad c>0 \\ -\dfrac{1}{\sqrt{c}}\cdot \operatorname{arsinh}\left(\dfrac{bx+2c}{\sqrt{\Delta}\,x}\right) & \text{für} \quad c>0,\ \Delta>0 \\ \dfrac{1}{\sqrt{|c|}}\cdot \arcsin\left(\dfrac{bx+2c}{\sqrt{|\Delta|}\,x}\right) & \text{für} \quad c<0,\ \Delta<0 \end{cases}$$

(198) $$\int \frac{dx}{x^2\sqrt{ax^2+bx+c}} = -\frac{\sqrt{ax^2+bx+c}}{cx} - \frac{b}{2c}\cdot \underbrace{\int \frac{dx}{x\sqrt{ax^2+bx+c}}}_{\text{Integral (197)}}$$

(199) $$\int \sqrt{(ax^2+bx+c)^3}\,dx = \frac{2ax+b}{8a}\sqrt{(ax^2+bx+c)^3} + \frac{3\Delta}{16a}\cdot \underbrace{\int \sqrt{ax^2+bx+c}\,dx}_{\text{Integral (189)}}$$

(200) $$\int x\sqrt{(ax^2+bx+c)^3}\,dx = \frac{1}{5a}\sqrt{(ax^2+bx+c)^5} - \frac{b}{2a}\cdot \underbrace{\int \sqrt{(ax^2+bx+c)^3}\,dx}_{\text{Integral (199)}}$$

(201) $$\int \frac{dx}{\sqrt{(ax^2+bx+c)^3}} = \frac{4ax+2b}{\Delta\sqrt{ax^2+bx+c}}$$

(202) $$\int \frac{x\,dx}{\sqrt{(ax^2+bx+c)^3}} = -\frac{2bx+4c}{\Delta\sqrt{ax^2+bx+c}}$$

(203) $$\int \frac{dx}{x\sqrt{(ax^2+bx+c)^3}} = \frac{1}{c\sqrt{ax^2+bx+c}} + \frac{1}{c}\cdot \underbrace{\int \frac{dx}{x\sqrt{ax^2+bx+c}}}_{\text{Integral (197)}} - \frac{b}{2c}\cdot \underbrace{\int \frac{dx}{\sqrt{(ax^2+bx+c)^3}}}_{\text{Integral (201)}}$$

$(c \neq 0)$

16 Integrale mit $\sin(ax)$ $(a \neq 0)$

Hinweis: Integrale mit einer Sinusfunktion *und* einer
- *Kosinusfunktion:* siehe Abschnitt 18
- *Exponentialfunktion:* siehe Abschnitt 22
- *Hyperbelfunktion:* siehe Abschnitt 24 und 25

(204) $$\int \sin(ax)\,dx = -\frac{\cos(ax)}{a}$$

(205) $$\int \sin^2(ax)\,dx = \frac{x}{2} - \frac{\sin(2ax)}{4a} = \frac{x}{2} - \frac{\sin(ax)\cdot\cos(ax)}{2a}$$

(206) $$\int \sin^3(ax)\,dx = -\frac{\cos(ax)}{a} + \frac{\cos^3(ax)}{3a}$$

(207) $$\int \sin^n(ax)\,dx = -\frac{\sin^{n-1}(ax)\cdot\cos(ax)}{na} + \frac{n-1}{n}\cdot\int \sin^{n-2}(ax)\,dx \qquad (n \neq 0)$$

(208) $$\int x\cdot\sin(ax)\,dx = \frac{\sin(ax)}{a^2} - \frac{x\cdot\cos(ax)}{a}$$

(209) $$\int x^2\cdot\sin(ax)\,dx = \frac{2x\cdot\sin(ax)}{a^2} - \frac{(a^2x^2-2)\cdot\cos(ax)}{a^3}$$

(210) $$\int x^n\cdot\sin(ax)\,dx = -\frac{x^n\cdot\cos(ax)}{a} + \frac{n\cdot x^{n-1}\cdot\sin(ax)}{a^2} - \frac{n(n-1)}{a^2}\cdot\int x^{n-2}\cdot\sin(ax)\,dx$$

$(n \geq 2)$

(211) $$\int \frac{\sin(ax)}{x}\,dx = ax - \frac{(ax)^3}{3\cdot 3!} + \frac{(ax)^5}{5\cdot 5!} - + \ldots$$

(Potenzreihenentwicklung: Konvergenz für $|x| < \infty$)

(212) $$\int \frac{\sin(ax)}{x^2}\,dx = -\frac{\sin(ax)}{x} + a\cdot\underbrace{\int \frac{\cos(ax)}{x}\,dx}_{\text{Integral (235)}}$$

(213) $$\int \frac{\sin(ax)}{x^n}\,dx = -\frac{\sin(ax)}{(n-1)\,x^{n-1}} + \frac{a}{n-1}\cdot\underbrace{\int \frac{\cos(ax)}{x^{n-1}}\,dx}_{\text{Integral (237)}} \qquad (n \neq 1)$$

Fall $n = 1$: siehe Integral (211)

(214) $$\int \frac{dx}{\sin(ax)} = \frac{1}{a}\cdot\ln\left|\tan\left(\frac{ax}{2}\right)\right|$$

(215)	$\int \frac{dx}{\sin^2(ax)} = -\frac{\cot(ax)}{a}$
(216)	$\int \frac{dx}{\sin^n(ax)} = -\frac{\cos(ax)}{a(n-1)\cdot\sin^{n-1}(ax)} + \frac{n-2}{n-1}\cdot\int \frac{dx}{\sin^{n-2}(ax)} \qquad (n > 1)$ Fall $n = 1$: siehe Integral (214)
(217)	$\int x\cdot\sin^2(ax)\,dx = \frac{x^2}{4} - \frac{x\cdot\sin(2ax)}{4a} - \frac{\cos(2ax)}{8a^2}$
(218)	$\int \frac{x\,dx}{\sin^2(ax)} = -\frac{x\cdot\cot(ax)}{a} + \frac{1}{a^2}\cdot\ln\lvert\sin(ax)\rvert$
(219)	$\int \frac{dx}{1\pm\sin(ax)} = \mp\frac{1}{a}\cdot\tan\left(\frac{\pi}{4}\mp\frac{ax}{2}\right)$
(220)	$\int \frac{dx}{p+q\cdot\sin(ax)} = \begin{cases} \frac{2}{a\sqrt{p^2-q^2}}\cdot\arctan\left(\frac{p\cdot\tan(ax/2)+q}{\sqrt{p^2-q^2}}\right) & \text{für } p^2 > q^2 \\ \frac{1}{a\sqrt{q^2-p^2}}\cdot\ln\left\lvert\frac{p\cdot\tan(ax/2)+q-\sqrt{q^2-p^2}}{p\cdot\tan(ax/2)+q+\sqrt{q^2-p^2}}\right\rvert & \text{für } p^2 < q^2 \end{cases}$ Fall $p^2 = q^2$: siehe Integral (219)
(221)	$\int \frac{x\,dx}{1+\sin(ax)} = -\frac{x}{a}\cdot\tan\left(\frac{\pi}{4}-\frac{ax}{2}\right) + \frac{2}{a^2}\cdot\ln\left\lvert\cos\left(\frac{\pi}{4}-\frac{ax}{2}\right)\right\rvert$
(222)	$\int \frac{x\,dx}{1-\sin(ax)} = \frac{x}{a}\cdot\cot\left(\frac{\pi}{4}-\frac{ax}{2}\right) + \frac{2}{a^2}\cdot\ln\left\lvert\sin\left(\frac{\pi}{4}-\frac{ax}{2}\right)\right\rvert$
(223)	$\int \frac{\sin(ax)\,dx}{1\pm\sin(ax)} = \pm x + \frac{1}{a}\cdot\tan\left(\frac{\pi}{4}\mp\frac{ax}{2}\right)$
(224)	$\int \frac{\sin(ax)\,dx}{p+q\cdot\sin(ax)} = \frac{x}{q} - \frac{p}{q}\cdot\underbrace{\int\frac{dx}{p+q\cdot\sin(ax)}}_{\text{Integral (220)}} \qquad (q \neq 0)$
(225)	$\int \frac{dx}{\sin(ax)\,[p+q\cdot\sin(ax)]} = \frac{1}{ap}\cdot\ln\left\lvert\tan\left(\frac{ax}{2}\right)\right\rvert - \frac{q}{p}\cdot\underbrace{\int\frac{dx}{p+q\cdot\sin(ax)}}_{\text{Integral (220)}} \qquad (p \neq 0)$
(226)	$\int \sin(ax)\cdot\sin(bx)\,dx = \frac{\sin((a-b)x)}{2(a-b)} - \frac{\sin((a+b)x)}{2(a+b)} \qquad (a^2 \neq b^2)$ Fall $a^2 = b^2$: siehe Integral (205)
(227)	$\int \sin(ax)\cdot\sin(ax+b)\,dx = -\frac{1}{4a}\cdot\sin(2ax+b) + \frac{(\cos b)}{2}\,x$

17 Integrale mit $\cos(ax)$ $(a \neq 0)$

Hinweis: Integrale mit einer Kosinusfunktion *und* einer
– *Sinusfunktion:* siehe Abschnitt 18
– *Exponentialfunktion:* siehe Abschnitt 22
– *Hyperbelfunktion:* siehe Abschnitt 24 und 25

(228) $$\int \cos(ax)\,dx = \frac{\sin(ax)}{a}$$

(229) $$\int \cos^2(ax)\,dx = \frac{x}{2} + \frac{\sin(2ax)}{4a} = \frac{x}{2} + \frac{\sin(ax)\cdot\cos(ax)}{2a}$$

(230) $$\int \cos^3(ax)\,dx = \frac{\sin(ax)}{a} - \frac{\sin^3(ax)}{3a}$$

(231) $$\int \cos^n(ax)\,dx = \frac{\cos^{n-1}(ax)\cdot\sin(ax)}{na} + \frac{n-1}{n}\cdot\int \cos^{n-2}(ax)\,dx \qquad (n \neq 0)$$

(232) $$\int x\cdot\cos(ax)\,dx = \frac{\cos(ax)}{a^2} + \frac{x\cdot\sin(ax)}{a}$$

(233) $$\int x^2\cdot\cos(ax)\,dx = \frac{2x\cdot\cos(ax)}{a^2} + \frac{(a^2x^2-2)\cdot\sin(ax)}{a^3}$$

(234) $$\int x^n\cdot\cos(ax)\,dx = \frac{x^n\cdot\sin(ax)}{a} + \frac{n\cdot x^{n-1}\cdot\cos(ax)}{a^2} - \frac{n(n-1)}{a^2}\cdot\int x^{n-2}\cdot\cos(ax)\,dx$$

$(n \geq 2)$

(235) $$\int \frac{\cos(ax)}{x}\,dx = \ln|ax| - \frac{(ax)^2}{2\cdot 2!} + \frac{(ax)^4}{4\cdot 4!} - \frac{(ax)^6}{6\cdot 6!} + - \ldots$$

(Potenzreihenentwicklung: Konvergenz für $|x| > 0$)

(236) $$\int \frac{\cos(ax)}{x^2}\,dx = -\frac{\cos(ax)}{x} - a\cdot\underbrace{\int \frac{\sin(ax)}{x}\,dx}_{\text{Integral (211)}}$$

(237) $$\int \frac{\cos(ax)}{x^n}\,dx = -\frac{\cos(ax)}{(n-1)\,x^{n-1}} - \frac{a}{n-1}\cdot\underbrace{\int \frac{\sin(ax)}{x^{n-1}}\,dx}_{\text{Integral (213)}} \qquad (n \neq 1)$$

Fall $n = 1$: siehe Integral (235)

(238) $$\int \frac{dx}{\cos(ax)} = \frac{1}{a}\cdot\ln\left|\tan\left(\frac{ax}{2} + \frac{\pi}{4}\right)\right|$$

(239) $$\int \frac{dx}{\cos^2(ax)} = \frac{\tan(ax)}{a}$$

(240) $$\int \frac{dx}{\cos^n(ax)} = \frac{\sin(ax)}{a(n-1)\cdot\cos^{n-1}(ax)} + \frac{n-2}{n-1}\cdot\int\frac{dx}{\cos^{n-2}(ax)} \qquad (n>1)$$

Fall $n = 1$: siehe Integral (238)

(241) $$\int x\cdot\cos^2(ax)\,dx = \frac{x^2}{4} + \frac{x\cdot\sin(2ax)}{4a} + \frac{\cos(2ax)}{8a^2}$$

(242) $$\int \frac{x\,dx}{\cos^2(ax)} = \frac{x\cdot\tan(ax)}{a} + \frac{1}{a^2}\cdot\ln|\cos(ax)|$$

(243) $$\int \frac{dx}{1+\cos(ax)} = \frac{1}{a}\cdot\tan\left(\frac{ax}{2}\right)$$

(244) $$\int \frac{dx}{1-\cos(ax)} = -\frac{1}{a}\cdot\cot\left(\frac{ax}{2}\right)$$

(245) $$\int \frac{dx}{p+q\cdot\cos(ax)} = \begin{cases} \dfrac{2}{a\sqrt{p^2-q^2}}\cdot\arctan\left(\dfrac{(p-q)\cdot\tan(ax/2)}{\sqrt{p^2-q^2}}\right) & \text{für} \quad p^2 > q^2 \\ \dfrac{1}{a\sqrt{q^2-p^2}}\cdot\ln\left|\dfrac{(q-p)\cdot\tan(ax/2)+\sqrt{q^2-p^2}}{(q-p)\cdot\tan(ax/2)-\sqrt{q^2-p^2}}\right| & \text{für} \quad p^2 < q^2 \end{cases}$$

Fall $p^2 = q^2$: siehe Integral (243) bzw. Integral (244)

(246) $$\int \frac{x\,dx}{1+\cos(ax)} = \frac{x}{a}\cdot\tan\left(\frac{ax}{2}\right) + \frac{2}{a^2}\cdot\ln\left|\cos\left(\frac{ax}{2}\right)\right|$$

(247) $$\int \frac{x\,dx}{1-\cos(ax)} = -\frac{x}{a}\cdot\cot\left(\frac{ax}{2}\right) + \frac{2}{a^2}\cdot\ln\left|\sin\left(\frac{ax}{2}\right)\right|$$

(248) $$\int \frac{\cos(ax)\,dx}{1+\cos(ax)} = x - \frac{1}{a}\cdot\tan\left(\frac{ax}{2}\right)$$

(249) $$\int \frac{\cos(ax)\,dx}{1-\cos(ax)} = -x - \frac{1}{a}\cdot\cot\left(\frac{ax}{2}\right)$$

(250) $$\int \frac{\cos(ax)\,dx}{p+q\cdot\cos(ax)} = \frac{x}{q} - \frac{p}{q}\cdot\underbrace{\int\frac{dx}{p+q\cdot\cos(ax)}}_{\text{Integral (245)}} \qquad (q \neq 0)$$

(251) $$\int \frac{dx}{\cos(ax)\,[p+q\cdot\cos(ax)]} = \frac{1}{ap}\cdot\ln\left|\tan\left(\frac{ax}{2}+\frac{\pi}{4}\right)\right| - \frac{q}{p}\cdot\underbrace{\int\frac{dx}{p+q\cdot\cos(ax)}}_{\text{Integral (245)}} \qquad (p \neq 0)$$

(252) $$\int \cos(ax)\cdot\cos(bx)\,dx = \frac{\sin((a-b)x)}{2(a-b)} + \frac{\sin((a+b)x)}{2(a+b)} \qquad (a^2 \neq b^2)$$

Fall $a^2 = b^2$: siehe Integral (229)

(253) $$\int \cos(ax)\cdot\cos(ax+b)\,dx = \frac{1}{4a}\cdot\sin(2ax+b) + \frac{(\cos b)}{2}\,x$$

18 Integrale mit $\sin(ax)$ und $\cos(ax)$ $(a \neq 0)$

(254) $$\int \sin(ax) \cdot \cos(ax)\,dx = \frac{\sin^2(ax)}{2a} = -\frac{1}{4a} \cdot \cos(2ax)$$

(255) $$\int \sin^n(ax) \cdot \cos(ax)\,dx = \frac{\sin^{n+1}(ax)}{(n+1)\,a} \qquad (n \neq -1)$$

Fall $n = -1$: siehe Integral (293)

(256) $$\int \sin(ax) \cdot \cos^n(ax)\,dx = -\frac{\cos^{n+1}(ax)}{(n+1)\,a} \qquad (n \neq -1)$$

Fall $n = -1$: siehe Integral (286)

(257) $$\int \sin^2(ax) \cdot \cos^2(ax)\,dx = \frac{x}{8} - \frac{\sin(4ax)}{32a}$$

(258) $$\int \sin^m(ax) \cdot \cos^n(ax)\,dx =$$

$$= \begin{cases} -\dfrac{\sin^{m-1}(ax) \cdot \cos^{(n+1)}(ax)}{(m+n)\,a} + \dfrac{m-1}{m+n} \cdot \displaystyle\int \sin^{m-2}(ax) \cdot \cos^n(ax)\,dx \\[2ex] \dfrac{\sin^{m+1}(ax) \cdot \cos^{(n-1)}(ax)}{(m+n)\,a} + \dfrac{n-1}{m+n} \cdot \displaystyle\int \sin^m(ax) \cdot \cos^{n-2}(ax)\,dx \end{cases}$$

Beide Formeln gelten nur für $m \neq -n$. Fall $m = -n$: siehe Integral (289) bzw. (296)

(259) $$\int \frac{dx}{\sin(ax) \cdot \cos(ax)} = \frac{1}{a} \cdot \ln|\tan(ax)|$$

(260) $$\int \frac{dx}{\sin^2(ax) \cdot \cos(ax)} = \frac{1}{a}\left[\ln\left|\tan\left(\frac{ax}{2} + \frac{\pi}{4}\right)\right| - \frac{1}{\sin(ax)}\right]$$

(261) $$\int \frac{dx}{\sin^m(ax) \cdot \cos(ax)} = -\frac{1}{(m-1)\,a \cdot \sin^{m-1}(ax)} + \int \frac{dx}{\sin^{m-2}(ax) \cdot \cos(ax)} \qquad (m \neq 1)$$

Fall $m = 1$: siehe Integral (259)

(262) $$\int \frac{dx}{\sin(ax) \cdot \cos^2(ax)} = \frac{1}{a}\left[\ln\left|\tan\left(\frac{ax}{2}\right)\right| + \frac{1}{\cos(ax)}\right]$$

(263) $$\int \frac{dx}{\sin(ax) \cdot \cos^n(ax)} = \frac{1}{(n-1)\,a \cdot \cos^{n-1}(ax)} + \int \frac{dx}{\sin(ax) \cdot \cos^{n-2}(ax)} \qquad (n \neq 1)$$

Fall $n = 1$: siehe Integral (259)

(264) $$\int \frac{dx}{\sin^m(ax)\cdot\cos^n(ax)} = $$
$$= \begin{cases} \dfrac{1}{(n-1)\,a\cdot\sin^{m-1}(ax)\cdot\cos^{n-1}(ax)} + \dfrac{m+n-2}{n-1}\cdot\displaystyle\int\frac{dx}{\sin^m(ax)\cdot\cos^{n-2}(ax)} \\ -\dfrac{1}{(m-1)\,a\cdot\sin^{m-1}(ax)\cdot\cos^{n-1}(ax)} + \dfrac{m+n-2}{m-1}\cdot\displaystyle\int\frac{dx}{\sin^{m-2}(ax)\cdot\cos^n(ax)} \end{cases}$$

Obere Formel für $n \neq 1$, untere Formel für $m \neq 1$.

Fall $n = 1$: siehe Integral (261); Fall $m = 1$: siehe Integral (263)

(265) $$\int \frac{\sin(ax)}{\cos(ax)}\,dx = \int \tan(ax)\,dx = -\frac{1}{a}\cdot\ln|\cos(ax)|$$

(266) $$\int \frac{\sin^2(ax)}{\cos(ax)}\,dx = -\frac{\sin(ax)}{a} + \frac{1}{a}\cdot\ln\left|\tan\left(\frac{ax}{2}+\frac{\pi}{4}\right)\right|$$

(267) $$\int \frac{\sin^m(ax)}{\cos(ax)}\,dx = -\frac{\sin^{m-1}(ax)}{(m-1)\,a} + \int\frac{\sin^{m-2}(ax)}{\cos(ax)}\,dx \qquad (m \neq 1)$$

Fall $m = 1$: siehe Integral (265)

(268) $$\int \frac{\sin(ax)}{\cos^2(ax)}\,dx = \frac{1}{a\cdot\cos(ax)}$$

(269) $$\int \frac{\sin(ax)}{\cos^n(ax)}\,dx = \frac{1}{(n-1)\,a\cdot\cos^{n-1}(ax)} \qquad (n \neq 1)$$

Fall $n = 1$: siehe Integral (265)

(270) $$\int \frac{\sin^2(ax)}{\cos^2(ax)}\,dx = \int \tan^2(ax)\,dx = \frac{\tan(ax)}{a} - x$$

(271) $$\int \frac{\sin^m(ax)}{\cos^n(ax)}\,dx = \begin{cases} \dfrac{\sin^{m-1}(ax)}{(n-1)\,a\cdot\cos^{n-1}(ax)} - \dfrac{m-1}{n-1}\cdot\displaystyle\int\frac{\sin^{m-2}(ax)}{\cos^{n-2}(ax)}\,dx & (n \neq 1) \\ \dfrac{\sin^{m+1}(ax)}{(n-1)\,a\cdot\cos^{n-1}(ax)} - \dfrac{m-n+2}{n-1}\cdot\displaystyle\int\frac{\sin^m(ax)}{\cos^{n-2}(ax)}\,dx & (n \neq 1) \\ -\dfrac{\sin^{m-1}(ax)}{(m-n)\,a\cdot\cos^{n-1}(ax)} + \dfrac{m-1}{m-n}\cdot\displaystyle\int\frac{\sin^{m-2}(ax)}{\cos^n(ax)} & (m \neq n) \end{cases}$$

Fall $n = 1$: siehe Integral (267); Fall $m = n$: siehe Integral (289)

(272) $$\int \frac{\cos(ax)}{\sin(ax)}\,dx = \int \cot(ax)\,dx = \frac{1}{a}\cdot\ln|\sin(ax)|$$

(273) $$\int \frac{\cos(ax)}{\sin^2(ax)}\,dx = -\frac{1}{a\cdot\sin(ax)}$$

(274) $$\int \frac{\cos(ax)}{\sin^n(ax)}\,dx = -\frac{1}{(n-1)\,a\cdot\sin^{n-1}(ax)} \qquad (n \neq 1)$$

Fall $n = 1$: siehe Integral (272) und (293)

(275) $$\int \frac{\cos^2(ax)}{\sin(ax)}\,dx = \frac{1}{a}\left[\cos(ax) + \ln\left|\tan\left(\frac{ax}{2}\right)\right|\right]$$

(276) $$\int \frac{\cos^m(ax)}{\sin(ax)}\,dx = \frac{\cos^{m-1}(ax)}{(m-1)\,a} + \int \frac{\cos^{m-2}(ax)}{\sin(ax)}\,dx \qquad (m \neq 1)$$

Fall $m = 1$: siehe Integral (272) und (293)

(277) $$\int \frac{\cos^m(ax)}{\sin^n(ax)}\,dx = \begin{cases} -\dfrac{\cos^{m-1}(ax)}{(n-1)\,a\cdot\sin^{n-1}(ax)} - \dfrac{m-1}{n-1}\cdot\displaystyle\int \frac{\cos^{m-2}(ax)}{\sin^{n-2}(ax)}\,dx & (n \neq 1) \\ -\dfrac{\cos^{m+1}(ax)}{(n-1)\,a\cdot\sin^{n-1}(ax)} - \dfrac{m-n+2}{n-1}\cdot\displaystyle\int \frac{\cos^{m}(ax)}{\sin^{n-2}(ax)}\,dx & (n \neq 1) \\ \dfrac{\cos^{m-1}(ax)}{(m-n)\,a\cdot\sin^{n-1}(ax)} + \dfrac{m-1}{m-n}\cdot\displaystyle\int \frac{\cos^{m-2}(ax)}{\sin^{n}(ax)} & (m \neq n) \end{cases}$$

Fall $n = 1$: siehe Integral (276); Fall $m = n$: siehe Integral (296)

(278) $$\int \frac{dx}{\sin(ax) \pm \cos(ax)} = \frac{1}{a\sqrt{2}}\cdot\ln\left|\tan\left(\frac{ax}{2} \pm \frac{\pi}{8}\right)\right|$$

(279) $$\int \frac{\sin(ax)\,dx}{\sin(ax) \pm \cos(ax)} = \frac{x}{2} \mp \frac{1}{2a}\cdot\ln|\sin(ax) \pm \cos(ax)|$$

(280) $$\int \frac{\cos(ax)\,dx}{\sin(ax) \pm \cos(ax)} = \pm\frac{x}{2} + \frac{1}{2a}\cdot\ln|\sin(ax) \pm \cos(ax)|$$

(281) $$\int \frac{dx}{\sin(ax)\,[1 \pm \cos(ax)]} = \pm\frac{1}{2a\,[1 \pm \cos(ax)]} + \frac{1}{2a}\cdot\ln\left|\tan\left(\frac{ax}{2}\right)\right|$$

(282) $$\int \frac{dx}{\cos(ax)\,[1 \pm \sin(ax)]} = \mp\frac{1}{2a\,[1 \pm \sin(ax)]} + \frac{1}{2a}\cdot\ln\left|\tan\left(\frac{ax}{2} + \frac{\pi}{4}\right)\right|$$

(283) $$\int \frac{\sin(ax)\,dx}{\cos(ax)\,[1 \pm \cos(ax)]} = \frac{1}{a}\cdot\ln\left|\frac{1 \pm \cos(ax)}{\cos(ax)}\right|$$

(284) $$\int \frac{\cos(ax)\,dx}{\sin(ax)\,[1 \pm \sin(ax)]} = -\frac{1}{a}\cdot\ln\left|\frac{1 \pm \sin(ax)}{\sin(ax)}\right|$$

(285) $$\int \sin(ax)\cdot\cos(bx)\,dx = -\frac{\cos((a+b)\,x)}{2\,(a+b)} - \frac{\cos((a-b)\,x)}{2\,(a-b)} \qquad (a^2 \neq b^2)$$

Fall $a^2 = b^2$: siehe Integral (254)

19 Integrale mit $\tan(ax)$ $(a \neq 0)$

(286) $\int \tan(ax)\,dx = -\frac{1}{a} \cdot \ln|\cos(ax)|$

(287) $\int \tan^2(ax)\,dx = \frac{\tan(ax)}{a} - x$

(288) $\int \tan^3(ax)\,dx = \frac{\tan^2(ax)}{2a} + \frac{1}{a} \cdot \ln|\cos(ax)|$

(289) $\int \tan^n(ax)\,dx = \frac{\tan^{n-1}(ax)}{(n-1)\,a} - \int \tan^{n-2}(ax)\,dx \qquad (n \neq 1)$

Fall $n = 1$: siehe Integral (286)

(290) $\int \frac{dx}{\tan(ax)} = \int \cot(ax)\,dx = \frac{1}{a} \cdot \ln|\sin(ax)|$

(291) $\int \frac{\tan^n(ax)}{\cos^2(ax)}\,dx = \frac{\tan^{n+1}(ax)}{(n+1)\,a} \qquad (n \neq -1)$

Fall $n = -1$: siehe Integral (259)

(292) $\int \frac{dx}{p + q \cdot \tan(ax)} = \frac{apx + q \cdot \ln|q \cdot \sin(ax) + p \cdot \cos(ax)|}{a(p^2 + q^2)} \qquad (q \neq 0)$

20 Integrale mit $\cot(ax)$ $(a \neq 0)$

(293) $\int \cot(ax)\,dx = \frac{1}{a} \cdot \ln|\sin(ax)|$

(294) $\int \cot^2(ax)\,dx = -\frac{\cot(ax)}{a} - x$

(295) $\int \cot^3(ax)\,dx = -\frac{\cot^2(ax)}{2a} - \frac{1}{a} \cdot \ln|\sin(ax)|$

(296) $\int \cot^n(ax)\,dx = -\frac{\cot^{n-1}(ax)}{(n-1)\,a} - \int \cot^{n-2}(ax)\,dx \qquad (n \neq 1)$

Fall $n = 1$: siehe Integral (293)

(297) $\displaystyle\int \frac{dx}{\cot(ax)} = \int \tan(ax)\,dx = -\frac{1}{a} \cdot \ln|\cos(ax)|$

(298) $\displaystyle\int \frac{\cot^n(ax)}{\sin^2(ax)}\,dx = -\frac{\cot^{n+1}(ax)}{(n+1)\,a} \qquad (n \neq -1)$

Fall $n = -1$: siehe Integral (259)

(299) $\displaystyle\int \frac{dx}{p + q \cdot \cot(ax)} = \frac{apx - q \cdot \ln|p \cdot \sin(ax) + q \cdot \cos(ax)|}{a(p^2 + q^2)} \qquad (q \neq 0)$

21 Integrale mit einer Arkusfunktion $(a \neq 0)$

(300) $\displaystyle\int \arcsin\left(\frac{x}{a}\right) dx = x \cdot \arcsin\left(\frac{x}{a}\right) + \sqrt{a^2 - x^2}$

(301) $\displaystyle\int x \cdot \arcsin\left(\frac{x}{a}\right) dx = \left(\frac{2x^2 - a^2}{4}\right) \cdot \arcsin\left(\frac{x}{a}\right) + \frac{x}{4} \cdot \sqrt{a^2 - x^2}$

(302) $\displaystyle\int x^2 \cdot \arcsin\left(\frac{x}{a}\right) dx = \frac{x^3}{3} \cdot \arcsin\left(\frac{x}{a}\right) + \left(\frac{x^2 + 2a^2}{9}\right) \cdot \sqrt{a^2 - x^2}$

(303) $\displaystyle\int \arccos\left(\frac{x}{a}\right) dx = x \cdot \arccos\left(\frac{x}{a}\right) - \sqrt{a^2 - x^2}$

(304) $\displaystyle\int x \cdot \arccos\left(\frac{x}{a}\right) dx = \left(\frac{2x^2 - a^2}{4}\right) \cdot \arccos\left(\frac{x}{a}\right) - \frac{x}{4} \cdot \sqrt{a^2 - x^2}$

(305) $\displaystyle\int x^2 \cdot \arccos\left(\frac{x}{a}\right) dx = \frac{x^3}{3} \cdot \arccos\left(\frac{x}{a}\right) - \left(\frac{x^2 + 2a^2}{9}\right) \cdot \sqrt{a^2 - x^2}$

(306) $\displaystyle\int \arctan\left(\frac{x}{a}\right) dx = x \cdot \arctan\left(\frac{x}{a}\right) - \frac{a}{2} \cdot \ln(x^2 + a^2)$

(307) $\displaystyle\int x \cdot \arctan\left(\frac{x}{a}\right) dx = \frac{1}{2}(x^2 + a^2) \cdot \arctan\left(\frac{x}{a}\right) - \frac{ax}{2}$

(308) $\displaystyle\int x^2 \cdot \arctan\left(\frac{x}{a}\right) dx = \frac{x^3}{3} \cdot \arctan\left(\frac{x}{a}\right) - \frac{ax^2}{6} + \frac{a^3}{6} \cdot \ln(x^2 + a^2)$

(309) $\displaystyle\int \operatorname{arccot}\left(\frac{x}{a}\right) dx = x \cdot \operatorname{arccot}\left(\frac{x}{a}\right) + \frac{a}{2} \cdot \ln(x^2 + a^2)$

(310) $\displaystyle\int x \cdot \operatorname{arccot}\left(\frac{x}{a}\right) dx = \frac{1}{2}(x^2 + a^2) \cdot \operatorname{arccot}\left(\frac{x}{a}\right) + \frac{ax}{2}$

(311) $\displaystyle\int x^2 \cdot \operatorname{arccot}\left(\frac{x}{a}\right) dx = \frac{x^3}{3} \cdot \operatorname{arccot}\left(\frac{x}{a}\right) + \frac{ax^2}{6} - \frac{a^3}{6} \cdot \ln(x^2 + a^2)$

22 Integrale mit e^{ax} $(a \neq 0)$

(312) $$\int e^{ax}\,dx = \frac{1}{a} \cdot e^{ax}$$

(313) $$\int x \cdot e^{ax}\,dx = \left(\frac{ax - 1}{a^2}\right) \cdot e^{ax}$$

(314) $$\int x^2 \cdot e^{ax}\,dx = \left(\frac{a^2x^2 - 2ax + 2}{a^3}\right) \cdot e^{ax}$$

(315) $$\int x^n \cdot e^{ax}\,dx = \frac{x^n \cdot e^{ax}}{a} - \frac{n}{a} \cdot \int x^{n-1} \cdot e^{ax}\,dx$$

(316) $$\int \frac{e^{ax}}{x}\,dx = \ln|ax| + \frac{ax}{1 \cdot 1!} + \frac{(ax)^2}{2 \cdot 2!} + \frac{(ax)^3}{3 \cdot 3!} + \ldots$$

(Potenzreihenentwicklung; Konvergenz für $|x| > 0$)

(317) $$\int \frac{e^{ax}}{x^n}\,dx = -\frac{e^{ax}}{(n-1)\,x^{n-1}} + \frac{a}{n-1} \cdot \int \frac{e^{ax}}{x^{n-1}}\,dx \qquad (n \neq 1)$$

Fall $n = 1$: siehe Integral (316)

(318) $$\int \frac{dx}{p + q \cdot e^{ax}} = \frac{x}{p} - \frac{1}{ap} \cdot \ln|p + q \cdot e^{ax}| \qquad (p \neq 0)$$

(319) $$\int \frac{e^{ax}\,dx}{p + q \cdot e^{ax}} = \frac{1}{aq} \cdot \ln|p + q \cdot e^{ax}| \qquad (q \neq 0)$$

(320) $$\int \frac{dx}{p \cdot e^{ax} + q \cdot e^{-ax}} = \begin{cases} \dfrac{1}{a\sqrt{pq}} \cdot \arctan\left(\sqrt{\dfrac{p}{q}} \cdot e^{ax}\right) & \text{für } pq > 0 \\[2ex] \dfrac{1}{2a\sqrt{|pq|}} \cdot \ln\left|\dfrac{q + \sqrt{|pq|} \cdot e^{ax}}{q - \sqrt{|pq|} \cdot e^{ax}}\right| & \text{für } pq < 0 \end{cases}$$

(321) $$\int e^{ax} \cdot \ln x\,dx = \frac{e^{ax} \cdot \ln|x|}{a} - \frac{1}{a} \cdot \underbrace{\int \frac{e^{ax}}{x}\,dx}_{\text{Integral (316)}}$$

(322) $$\int e^{ax} \cdot \sin(bx)\,dx = \frac{e^{ax}}{a^2 + b^2}\,[a \cdot \sin(bx) - b \cdot \cos(bx)]$$

(323) $$\int e^{ax} \cdot \sin^n(bx)\,dx = \frac{e^{ax} \cdot \sin^{n-1}(bx)}{a^2 + n^2b^2}\,[a \cdot \sin(bx) - nb \cdot \cos(bx)] + $$
$$+ \frac{n(n-1)\,b^2}{a^2 + n^2b^2} \cdot \int e^{ax} \cdot \sin^{n-2}(bx)\,dx$$

(324) $\int e^{ax} \cdot \cos(bx)\, dx = \frac{e^{ax}}{a^2 + b^2}\,[a \cdot \cos(bx) + b \cdot \sin(bx)]$

(325) $\int e^{ax} \cdot \cos^n(bx)\, dx = \frac{e^{ax} \cdot \cos^{n-1}(bx)}{a^2 + n^2 b^2}\,[a \cdot \cos(bx) + nb \cdot \sin(bx)] +$

$+ \frac{n(n-1)\, b^2}{a^2 + n^2 b^2} \cdot \int e^{ax} \cdot \cos^{n-2}(bx)\, dx$

(326) $\int e^{ax} \cdot \sinh(ax)\, dx = \frac{e^{2ax}}{4a} - \frac{x}{2}$

(327) $\int e^{ax} \cdot \sinh(bx)\, dx = \frac{e^{ax}}{a^2 - b^2}\,[a \cdot \sinh(bx) - b \cdot \cosh(bx)] \qquad (a^2 \neq b^2)$

Fall $a^2 = b^2$: siehe Integral (326)

(328) $\int e^{ax} \cdot \cosh(ax)\, dx = \frac{e^{2ax}}{4a} + \frac{x}{2}$

(329) $\int e^{ax} \cdot \cosh(bx)\, dx = \frac{e^{ax}}{a^2 - b^2}\,[a \cdot \cosh(bx) - b \cdot \sinh(bx)] \qquad (a^2 \neq b^2)$

Fall $a^2 = b^2$: siehe Integral (328)

(330) $\int x \cdot e^{ax} \cdot \sin(bx)\, dx = \frac{x \cdot e^{ax}}{a^2 + b^2}\,[a \cdot \sin(bx) - b \cdot \cos(bx)] -$

$- \frac{e^{ax}}{(a^2 + b^2)^2}\,[(a^2 - b^2) \cdot \sin(bx) - 2ab \cdot \cos(bx)]$

(331) $\int x \cdot e^{ax} \cdot \cos(bx)\, dx = \frac{x \cdot e^{ax}}{a^2 + b^2}\,[a \cdot \cos(bx) + b \cdot \sin(bx)] -$

$- \frac{e^{ax}}{(a^2 + b^2)^2}\,[(a^2 - b^2) \cdot \cos(bx) + 2ab \cdot \sin(bx)]$

23 Integrale mit $\ln x \quad (x > 0)$

Hinweis: Integrale mit einer Logarithmus- *und* einer Exponentialfunktion: siehe Abschnitt 22.

(332) $\int \ln x\, dx = x \cdot \ln x - x = x(\ln x - 1)$

(333) $\int (\ln x)^2\, dx = x(\ln x)^2 - 2x \cdot \ln x + 2x = x[(\ln x)^2 - 2 \cdot \ln x + 2)]$

(334) $\int (\ln x)^3\, dx = x(\ln x)^3 - 3x(\ln x)^2 + 6x \cdot \ln x - 6x = x[(\ln x)^3 - 3(\ln x)^2 + 6 \cdot \ln x - 6]$

(335)	$\int (\ln x)^n \, dx = x (\ln x)^n - n \cdot \int (\ln x)^{n-1} \, dx \qquad (n \neq -1)$ Fall $n = -1$: siehe Integral (336)
(336)	$\int \frac{dx}{\ln x} = \ln \lvert \ln x \rvert + \frac{\ln x}{1 \cdot 1!} + \frac{(\ln x)^2}{2 \cdot 2!} + \frac{(\ln x)^3}{3 \cdot 3!} + \ldots \qquad (x \neq 1)$
(337)	$\int x \cdot \ln x \, dx = \frac{1}{2} x^2 \left(\ln x - \frac{1}{2} \right)$
(338)	$\int x^2 \cdot \ln x \, dx = \frac{1}{3} x^3 \left(\ln x - \frac{1}{3} \right)$
(339)	$\int x^m \cdot \ln x \, dx = \frac{x^{m+1}}{m+1} \left(\ln x - \frac{1}{m+1} \right) \qquad (m \neq -1)$ Fall $m = -1$: siehe Integral (340)
(340)	$\int \frac{\ln x}{x} \, dx = \frac{1}{2} (\ln x)^2$
(341)	$\int \frac{\ln x}{x^m} \, dx = - \frac{\ln x}{(m-1) x^{m-1}} - \frac{1}{(m-1)^2 x^{m-1}} \qquad (m \neq 1)$ Fall $m = 1$: siehe Integral (340)
(342)	$\int \frac{(\ln x)^n}{x} \, dx = \frac{(\ln x)^{n+1}}{n+1} \qquad (n \neq -1)$ Fall $n = -1$: siehe Integral (343)
(343)	$\int \frac{dx}{x \cdot \ln x} = \ln \lvert \ln x \rvert \qquad (x \neq 1)$
(344)	$\int \frac{x^m}{\ln x} \, dx = \ln \lvert \ln x \rvert + (m+1) \ln x + \frac{(m+1)^2}{2 \cdot 2!} (\ln x)^2 + \frac{(m+1)^3}{3 \cdot 3!} (\ln x)^3 + \ldots \qquad (x \neq 1)$
(345)	$\int x^m \cdot (\ln x)^n \, dx = \frac{x^{m+1} \cdot (\ln x)^n}{m+1} - \frac{n}{m+1} \cdot \int x^m \cdot (\ln x)^{n-1} \, dx \qquad (m \neq -1)$ Fall $m = -1$: siehe Integral (342)
(346)	$\int \frac{x^m}{(\ln x)^n} \, dx = - \frac{x^{m+1}}{(n-1)(\ln x)^{n-1}} + \frac{m+1}{n-1} \cdot \int \frac{x^m}{(\ln x)^{n-1}} \, dx \qquad (n \neq 1;\ x \neq 1)$ Fall $n = 1$: siehe Integral (344)
(347)	$\int \ln (x^2 + a^2) \, dx = x \cdot \ln (x^2 + a^2) - 2x + 2a \cdot \arctan \left(\frac{x}{a} \right) \qquad (a \neq 0)$
(348)	$\int \ln (x^2 - a^2) \, dx = x \cdot \ln (x^2 - a^2) - 2x + a \cdot \ln \left(\frac{x+a}{x-a} \right) \qquad (x^2 > a^2)$

24 Integrale mit $\sinh(ax)$ $(a \neq 0)$

Hinweis: Integrale mit einer hyperbolischen Sinusfunktion *und* einer
– *hyperbolischen Kosinusfunktion:* siehe Abschnitt 26
– *Exponentialfunktion:* siehe Abschnitt 22

(349) $$\int \sinh(ax)\,dx = \frac{\cosh(ax)}{a}$$

(350) $$\int \sinh^2(ax)\,dx = \frac{\sinh(2ax)}{4a} - \frac{x}{2}$$

(351) $$\int \sinh^n(ax)\,dx = \frac{\sinh^{n-1}(ax)\cdot\cosh(ax)}{na} - \frac{n-1}{n}\cdot\int \sinh^{n-2}(ax)\,dx \qquad (n \neq 0)$$

(352) $$\int x\cdot\sinh(ax)\,dx = \frac{x\cdot\cosh(ax)}{a} - \frac{\sinh(ax)}{a^2}$$

(353) $$\int x^n\cdot\sinh(ax)\,dx = \frac{x^n\cdot\cosh(ax)}{a} - \frac{n\cdot x^{n-1}\cdot\sinh(ax)}{a^2} + \frac{n(n-1)}{a^2}\cdot\int x^{n-2}\cdot\sinh(ax)\,dx \qquad (n \geq 2)$$

(354) $$\int \frac{\sinh(ax)}{x}\,dx = ax + \frac{(ax)^3}{3\cdot 3!} + \frac{(ax)^5}{5\cdot 5!} + \ldots$$

(Potenzreihenentwicklung; Konvergenz für $|x| < \infty$)

(355) $$\int \frac{\sinh(ax)}{x^n}\,dx = -\frac{\sinh(ax)}{(n-1)\,x^{n-1}} + \frac{a}{n-1}\cdot\underbrace{\int \frac{\cosh(ax)}{x^{n-1}}\,dx}_{\text{Integral (369)}} \qquad (n \neq 1)$$

Fall $n = 1$: siehe Integral (354)

(356) $$\int \frac{dx}{\sinh(ax)} = \frac{1}{a}\cdot\ln\left|\tanh\left(\frac{ax}{2}\right)\right|$$

(357) $$\int \frac{dx}{\sinh^n(ax)} = -\frac{\cosh(ax)}{(n-1)\,a\cdot\sinh^{n-1}(ax)} - \frac{n-2}{n-1}\cdot\int \frac{dx}{\sinh^{n-2}(ax)} \qquad (n \neq 1)$$

Fall $n = 1$: siehe Integral (356)

(358) $$\int \frac{dx}{p + q\cdot\sinh(ax)} = \frac{1}{a\sqrt{p^2+q^2}}\cdot\ln\left|\frac{q\cdot e^{ax} + p - \sqrt{p^2+q^2}}{q\cdot e^{ax} + p + \sqrt{p^2+q^2}}\right| \qquad (q \neq 0)$$

(359) $$\int \frac{\sinh(ax)\,dx}{p + q\cdot\sinh(ax)} = \frac{x}{q} - \frac{p}{q}\cdot\underbrace{\int \frac{dx}{p + q\cdot\sinh(ax)}}_{\text{Integral (358)}} \qquad (q \neq 0)$$

(360) $\displaystyle\int \sinh(ax) \cdot \sinh(bx)\,dx = \frac{\sinh((a+b)x)}{2(a+b)} - \frac{\sinh((a-b)x)}{2(a-b)} \qquad (a^2 \neq b^2)$

Fall $a^2 = b^2$: siehe Integral (350)

(361) $\displaystyle\int \sinh(ax) \cdot \sin(bx)\,dx = \frac{a \cdot \cosh(ax) \cdot \sin(bx) - b \cdot \sinh(ax) \cdot \cos(bx)}{a^2 + b^2}$

(362) $\displaystyle\int \sinh(ax) \cdot \cos(bx)\,dx = \frac{a \cdot \cosh(ax) \cdot \cos(bx) + b \cdot \sinh(ax) \cdot \sin(bx)}{a^2 + b^2}$

25 Integrale mit $\cosh(ax) \quad (a \neq 0)$

Hinweis: Integrale mit einer hyperbolischen Kosinusfunktion *und* einer
– *hyperbolischen Sinusfunktion:* siehe Abschnitt 26
– *Exponentialfunktion:* siehe Abschnitt 22

(363) $\displaystyle\int \cosh(ax)\,dx = \frac{\sinh(ax)}{a}$

(364) $\displaystyle\int \cosh^2(ax)\,dx = \frac{\sinh(2ax)}{4a} + \frac{x}{2}$

(365) $\displaystyle\int \cosh^n(ax)\,dx = \frac{\cosh^{n-1}(ax) \cdot \sinh(ax)}{na} + \frac{n-1}{n} \cdot \int \cosh^{n-2}(ax)\,dx \qquad (n \neq 0)$

(366) $\displaystyle\int x \cdot \cosh(ax)\,dx = \frac{x \cdot \sinh(ax)}{a} - \frac{\cosh(ax)}{a^2}$

(367) $\displaystyle\int x^n \cdot \cosh(ax)\,dx = \frac{x^n \cdot \sinh(ax)}{a} - \frac{n \cdot x^{n-1} \cdot \cosh(ax)}{a^2} +$

$$+ \frac{n(n-1)}{a^2} \cdot \int x^{n-2} \cdot \cosh(ax)\,dx \qquad (n \geq 2)$$

(368) $\displaystyle\int \frac{\cosh(ax)}{x}\,dx = \ln|ax| + \frac{(ax)^2}{2 \cdot 2!} + \frac{(ax)^4}{4 \cdot 4!} + \frac{(ax)^6}{6 \cdot 6!} + \ldots$

(Potenzreihenentwicklung; Konvergenz für $|x| > 0$)

(369) $\displaystyle\int \frac{\cosh(ax)}{x^n}\,dx = -\frac{\cosh(ax)}{(n-1)\,x^{n-1}} + \frac{a}{n-1} \cdot \underbrace{\int \frac{\sinh(ax)}{x^{n-1}}\,dx}_{\text{Integral (355)}} \qquad (n \neq 1)$

Fall $n = 1$: siehe Integral (368)

(370) $\displaystyle\int \frac{dx}{\cosh(ax)} = \frac{2}{a} \cdot \arctan(e^{ax})$

(371) $$\int \frac{dx}{\cosh^n(ax)} = \frac{\sinh(ax)}{(n-1)\,a\cdot\cosh^{n-1}(ax)} + \frac{n-2}{n-1}\cdot\int\frac{dx}{\cosh^{n-2}(ax)} \qquad (n \neq 1)$$

Fall $n = 1$: siehe Integral (370)

(372) $$\int \frac{dx}{p + q\cdot\cosh(ax)} = \begin{cases} \dfrac{1}{a\sqrt{p^2 - q^2}}\cdot\ln\left|\dfrac{q\cdot e^{ax} + p - \sqrt{p^2-q^2}}{q\cdot e^{ax} + p + \sqrt{p^2-q^2}}\right| & \text{für } q > 0,\ p^2 > q^2 \\ \dfrac{-2}{a(p + q\cdot e^{ax})} & \text{für } p^2 = q^2 \neq 0 \\ \dfrac{2}{a\sqrt{q^2-p^2}}\cdot\arctan\left(\dfrac{p + q\cdot e^{ax}}{\sqrt{q^2-p^2}}\right) & \text{für } p^2 < q^2 \end{cases}$$

(373) $$\int \frac{\cosh(ax)\,dx}{p + q\cdot\cosh(ax)} = \frac{x}{q} - \frac{p}{q}\cdot\underbrace{\int\frac{dx}{p + q\cdot\cosh(ax)}}_{\text{Integral (372)}} \qquad (q \neq 0)$$

(374) $$\int \cosh(ax)\cdot\cosh(bx)\,dx = \frac{\sinh((a+b)\,x)}{2\,(a+b)} + \frac{\sinh((a-b)\,x)}{2\,(a-b)} \qquad (a^2 \neq b^2)$$

Fall $a^2 = b^2$: siehe Integral (364)

(375) $$\int \cosh(ax)\cdot\sin(bx)\,dx = \frac{a\cdot\sinh(ax)\cdot\sin(bx) - b\cdot\cosh(ax)\cdot\cos(bx)}{a^2 + b^2}$$

(376) $$\int \cosh(ax)\cdot\cos(bx)\,dx = \frac{a\cdot\sinh(ax)\cdot\cos(bx) + b\cdot\cosh(ax)\cdot\sin(bx)}{a^2 + b^2}$$

26 Integrale mit $\sinh(ax)$ und $\cosh(ax)$ $(a \neq 0)$

(377) $$\int \sinh(ax)\cdot\cosh(ax)\,dx = \frac{\sinh^2(ax)}{2a} = \frac{1}{4a}\cdot\cosh(2ax)$$

(378) $$\int \sinh(ax)\cdot\cosh(bx)\,dx = \frac{\cosh((a+b)\,x)}{2\,(a+b)} + \frac{\cosh((a-b)\,x)}{2\,(a-b)} \qquad (a^2 \neq b^2)$$

Fall $a^2 = b^2$: siehe Integral (377)

(379) $$\int \sinh^n(ax)\cdot\cosh(ax)\,dx = \frac{\sinh^{n+1}(ax)}{(n+1)\,a} \qquad (n \neq -1)$$

Fall $n = -1$: siehe Integral (384)

(380) $$\int \sinh(ax)\cdot\cosh^n(ax)\,dx = \frac{\cosh^{n+1}(ax)}{(n+1)\,a} \qquad (n \neq -1)$$

Fall $n = -1$: siehe Integral (382)

(381) $\int \sinh^2(ax) \cdot \cosh^2(ax)\,dx = \dfrac{\sinh(4ax)}{32a} - \dfrac{x}{8}$

(382) $\int \dfrac{\sinh(ax)}{\cosh(ax)}\,dx = \int \tanh(ax)\,dx = \dfrac{1}{a} \cdot \ln(\cosh(ax))$

(383) $\int \dfrac{\sinh^2(ax)}{\cosh(ax)}\,dx = \dfrac{\sinh(ax)}{a} - \dfrac{1}{a} \cdot \arctan(\sinh(ax))$

(384) $\int \dfrac{\cosh(ax)}{\sinh(ax)}\,dx = \int \coth(ax)\,dx = \dfrac{1}{a} \cdot \ln|\sinh(ax)|$

(385) $\int \dfrac{\cosh^2(ax)}{\sinh(ax)}\,dx = \dfrac{\cosh(ax)}{a} + \dfrac{1}{a} \cdot \ln\left|\tanh\left(\dfrac{ax}{2}\right)\right|$

(386) $\int \dfrac{dx}{\sinh(ax) \cdot \cosh(ax)} = \dfrac{1}{a} \cdot \ln|\tanh(ax)|$

27 Integrale mit $\tanh(ax) \quad (a \neq 0)$

(387) $\int \tanh(ax)\,dx = \dfrac{1}{a} \cdot \ln(\cosh(ax))$

(388) $\int \tanh^2(ax)\,dx = x - \dfrac{\tanh(ax)}{a}$

(389) $\int \tanh^n(ax)\,dx = -\dfrac{\tanh^{n-1}(ax)}{(n-1)\,a} + \int \tanh^{n-2}(ax)\,dx \qquad (n \neq 1)$

Fall $n = 1$: siehe Integral (387)

(390) $\int \dfrac{dx}{\tanh(ax)} = \int \coth(ax)\,dx = \dfrac{1}{a} \cdot \ln|\sinh(ax)|$

(391) $\int x \cdot \tanh^2(ax)\,dx = \dfrac{x^2}{2} - \dfrac{x \cdot \tanh(ax)}{a} + \dfrac{1}{a^2} \cdot \ln(\cosh(ax))$

28 Integrale mit $\coth(ax) \quad (a \neq 0)$

(392) $\int \coth(ax)\,dx = \dfrac{1}{a} \cdot \ln|\sinh(ax)|$

(393) $\int \coth^2(ax)\,dx = x - \dfrac{\coth(ax)}{a}$

(394)	$\int \coth^{n}(ax)\,dx = -\dfrac{\coth^{n-1}(ax)}{(n-1)\,a} + \int \coth^{n-2}(ax)\,dx \qquad (n \neq 1)$ Fall $n = 1$: siehe Integral (392)
(395)	$\int \dfrac{dx}{\coth(ax)} = \int \tanh(ax)\,dx = \dfrac{1}{a} \cdot \ln(\cosh(ax))$
(396)	$\int x \cdot \coth^{2}(ax)\,dx = \dfrac{x^{2}}{2} - \dfrac{x \cdot \coth(ax)}{a} + \dfrac{1}{a^{2}} \cdot \ln\lvert\sinh(ax)\rvert$

29 Integrale mit einer Areafunktion $(a \neq 0)$

(397)	$\int \operatorname{arsinh}\left(\dfrac{x}{a}\right) dx = x \cdot \operatorname{arsinh}\left(\dfrac{x}{a}\right) - \sqrt{x^{2} + a^{2}}$
(398)	$\int x \cdot \operatorname{arsinh}\left(\dfrac{x}{a}\right) dx = \left(\dfrac{2x^{2} + a^{2}}{4}\right) \cdot \operatorname{arsinh}\left(\dfrac{x}{a}\right) - \dfrac{x}{4} \cdot \sqrt{x^{2} + a^{2}}$
(399)	$\int \operatorname{arcosh}\left(\dfrac{x}{a}\right) dx = x \cdot \operatorname{arcosh}\left(\dfrac{x}{a}\right) - \sqrt{x^{2} - a^{2}}$
(400)	$\int x \cdot \operatorname{arcosh}\left(\dfrac{x}{a}\right) dx = \left(\dfrac{2x^{2} - a^{2}}{4}\right) \cdot \operatorname{arcosh}\left(\dfrac{x}{a}\right) - \dfrac{x}{4} \cdot \sqrt{x^{2} - a^{2}}$
(401)	$\int \operatorname{artanh}\left(\dfrac{x}{a}\right) dx = x \cdot \operatorname{artanh}\left(\dfrac{x}{a}\right) + \dfrac{a}{2} \cdot \ln\lvert a^{2} - x^{2}\rvert$
(402)	$\int x \cdot \operatorname{artanh}\left(\dfrac{x}{a}\right) dx = \dfrac{ax}{2} + \left(\dfrac{x^{2} - a^{2}}{2}\right) \cdot \operatorname{artanh}\left(\dfrac{x}{a}\right)$
(403)	$\int \operatorname{arcoth}\left(\dfrac{x}{a}\right) dx = x \cdot \operatorname{arcoth}\left(\dfrac{x}{a}\right) + \dfrac{a}{2} \cdot \ln\lvert x^{2} - a^{2}\rvert$
(404)	$\int x \cdot \operatorname{arcoth}\left(\dfrac{x}{a}\right) dx = \dfrac{ax}{2} + \left(\dfrac{x^{2} - a^{2}}{2}\right) \cdot \operatorname{arcoth}\left(\dfrac{x}{a}\right)$